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A Generalization of the Weddle Surface, of its Cremona Group, and of its Parametric Expression in Terms of Hyperelliptic Theta Functions.

By Arthur B. Coble.

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1. Introduction. The two surfaces of greatest interest which occur in the study of the hyperelliptic theta functions of genus two are the two quartic surfaces discovered respectively by Weddle and by Kummer. The simplest definition of the Kummer surface with 16 nodes is transcendental—the coördinates of a point on the surface are proportional to theta functions of order two and like characteristic. The simplest definition of the Weddle surface with 6 nodes is projective—the surface is the locus of nodes of quadrics on six given points. These are the nodes of the locus. The most natural extension of the Kummer surface or 2-way is to the Kummer p-way, a variety, $K_p^{m(p)}$, of order $m(p)$ in a linear space, $S_{\nu(p)}$, of dimension $\nu(p)$, where

$$\nu(p) = 2^p - 1, \qquad m(p) = p!\, 2^{p-1}. \tag{1}$$

The coördinates of a point on K_p are proportional to theta functions of order two, of like characteristic, and of genus p (i. e. functions of p variables). This extension has been made by Wirtinger (for references cf.[1] pp. 94-109). On the other hand a natural extension of the Weddle surface is not immediately apparent, either in its projective definition or in such transcendental definitions as have been given.

The author has indicated [2] very briefly an extension which is based on the connection between a point of the Weddle surface and a projectively canonical form of the plane curve of genus two, the nodal planar quartic curve. It is the purpose of this article to elaborate this indication, and to study the generalized Weddle manifold W_p in the linear space S_{2p-1}, in part for a general value of p, but with greater particularity for $p = 3$. The canonical plane curve is the hyperelliptic curve H_p^{p+2} of order $p+2$, genus p, and with a p-fold point (cf.[1] pp. 125-135). A fundamental difference between K_p and W_p is to be noted. The manifold K_p depends upon the $p(p+1)/2$ moduli of the general theta function while W_p depends upon the $2p-1$ moduli of the hyperelliptic theta functions which are particular cases of the abelian theta functions defined by the algebraic curve of genus p with

$3p-3$ moduli. Whether there exists a type of manifold W_p for the more general manifold K_p is an open question. It is certain however that the peculiarities of the hyperelliptic K_p are more easily studied as a birational transform of the W_p we are about to discuss than as a particular case of the K_p with general moduli.

The Weddle surface is invariant under an abelian Cremona group G_{32} of type (1, 1, 1, 1, 1) which contains a particular element, or G_2, for which every point of W_2 is invariant. The factor group of G_{32} with respect to this invariant G_2 is a g_{16}. The permutations of the points of W_2 under this g_{16} are precisely those effected by adding the 16 half-periods to the arguments u_1, u_2 of a point of the surface when the coördinates of the point are expressed in terms of theta functions. In § 1 we discuss the extension of this group, an abelian G_2^{n+2} of type $(1, 1, \cdots, 1)$ in a space S_n. For $n=2$ this group has been noted by Autonne ([3] p. 103); for $n=3$, by Kantor ([3] p. 213). The group is defined by $n+3$ points. *When n is odd,* it contains a particular element, or G_2, distinguished from the others by the fact that it involves all $n+3$ of the points symmetrically as F-points (fundamental or singular points) of the transformation. For $n>4$ this element, and the additional *regular transformation,*

$$x_i x_i' = 1 \qquad (i = 0, 1, \cdots, n), \tag{2}$$

are the only types of Cremona transformation defined symmetrically by a finite number of F-points ([4] II, p. 369). When $n=2p-1$ this element has a manifold of fixed points of dimension p *which we define to be* W_p. When W_p in S_{2p-1} is projected from one of the $2p+2$ F-points upon a W_p' in S_{2p-2}, the W_p' is invariant under the G_2^{n+2} of this S_n $(n=2p-2)$. The latter group is simply isomorphic with the group of the addition of the 2^{2p} half-periods of the functions of genus p.

In § 2 a point of W_p is related to a type of planar hyperelliptic curve H_p^{p+2} with p-fold point at O and branch points of its unique linear series g_1^2 at $r_1, \cdots, r_{2p+2}$. Projective conditions on the branch points and O, expressed in terms of theta functions, are translated to W_p to obtain the parametric expressions for the coördinates of a point on W_p in terms of theta functions, analogous to those given by Schottky for the Weddle surface ([5]; [1] pp. 122-124).

The projective situation in S_{2p-1} then implies the existence of certain three-term relations connecting theta products. Such relations are immediate consequences of the general theory only in the case of small values of p. A systematic application of available three-term relations to geometric mani-

folds has been made by Schottky. These comprise products of genus two, and the Weddle surface ([1] pp. 122-124); products of genus three with u_1, u_2, u_3 subject to $\vartheta(u)=0$, and Cayley's dianodal surface ([1] pp. 181-184); modular products (i. e., products of even thetas for zero arguments) of genus three, and the planar quartic curve ([1] pp. 173-177); modular products of genus four when $\vartheta(0)=0$, and the space sextic of genus four on a quadric cone ([1] pp. 268-271); and modular products of genus four, and the ten nodes of a Cayley symmetroid ([1] pp. 273-276). To these cases treated by Schottky we are adding in § 3 the hyperelliptic types. A mapping of W_p upon the hyperelliptic Kummer manifold K_p is immediately apparent.

In § 4 the particular case, $p=3$, is developed in greater detail. Properties of the manifolds, W_3 in S_5, and W_3' in S_4, of orders 19 and 10 respectively, are derived. On W_3' in S_4 there appears a notable surface of order 11 with seven triple points, which is intimately related to the Weddle surface.

§ 1

The Cremona $G_{2^{n+2}}$ in S_n.

2. The F-spaces of regular transformations in S_n. An F-point of a Cremona transformation of order m in S_n,

$$T: \qquad x_i' = \phi_i(x) = (\phi_i x)^m \qquad (i = 0, \cdots, n), \tag{3}$$

is a point f common to all the members, $\phi_i = 0$, which define the homaloidal system. The correspondent of f under the transformation is indeterminate. The manifold common to the loci, $\phi_i = 0$, must be made up of irreducible manifolds of various dimensions which are called the F-loci of T. These F-loci may meet in various loci of smaller dimension each of which is to be included among the F-loci of T. When we speak of a point on an F-locus of T it will be understood that the F-locus is irreducible and that the point is a generic point of it. No analysis of the nature and relative situation of these F-loci for transformations in higher space has been made and we pursue the matter only so far as is necessary for our immediate purpose.

Let f be any F-point, say a multiple point of order k on the generic member of the homaloidal system. It is a linear condition on the system that the members pass through f in such wise that a generic direction at f be on a member of the system. This linear condition on the system ϕ corresponds to a linear condition on the S_{n-1}'s, $x_i' = 0$, and is satisfied by all the S_{n-1}'s on a definite point x' which thus corresponds to a direction at f.

If y is a point on an S_{n-1} which is not on f and the direction at f is that of the line fy, the coördinates of the point x' are

$$x_i' = (\phi_i f)^{m-k}(\phi_i y)^k. \tag{4}$$

Thus as y runs over the ∞^{n-1} points of $S_{n-1}(y)$ the point x' may run over a manifold P of dimension $n-1$. In this case f is an *F-point of the first kind,* and P is the *principal manifold* or *P-locus of the first kind* which corresponds to it. The number of such F-points is necessarily finite. Otherwise two generic lines in the space x' would correspond to two algebraic curves in the space x which would have different directions at an infinite number of common points. These F-points of the first kind are in general isolated from each other, though naturally, in particular cases, either coalescences may occur, or they may be located on one or more F-loci of higher dimension. The P-locus is the map of the $S_{n-1}(y)$ by the linear system

$$\sum_{i=0}^{i=n} \lambda_i (\phi_i f)^{m-k}(\phi_i y)^k = 0.$$

It may happen however that the mapping just mentioned is singular in that as y runs over $S_{n-1}(y)$ the point x' runs over a manifold M of dimension $n-1-r$. Then f is an *F-point of the* $(r+1)$*-th kind* and the directions about f correspond only to the points of M. If f is generic on an F-locus, F_r, of dimension r, then, as f runs over F_r, either the manifold M may run over a system ∞^r which makes up a P-locus of dimension $n-1$ *of the* $(r+1)$*-th kind,* or the manifold M may remain fixed and itself constitute an F_{n-r-1}-locus of the inverse transformation T^{-1}. It is this latter case only which occurs in regular transformations. We disregard the possibility that M may vary within the limits ∞^r and ∞^0, and prove that

(5) *The regular transformations T in S_n have, and are determined by, a finite number of F-points of the first kind. They have also a finite number of F-loci of the* $(r+1)$*-th kind* $(r = 1, \cdots, n-2)$, *which are determined by the F-points of the first kind. An F_r-locus of the* $(r+1)$*-th kind of T is paired with an F_{n-r-1}-locus of the* $(n-r)$*-th kind of T^{-1} in such a way that the directions about each point of F_r correspond to all the points of F_{n-r-1} under T, and vice versa.*

A *regular transformation of order n* in S_n is by definition a transformation of the involutorial type (2) (when x, x' are in the same S_n) preceded or followed by a collineation. Hence the projective properties of this type can be deduced from the type (2). A *regular transformation* in S_n is a product of regular transformations of order n. In forming such regular pro-

ducts ST as we shall have occasion to use, the isolated F-points of T are placed in part at the isolated F-points of S^{-1} and, for the rest, in general position. The theory of the isolated F-points and the P-loci of the first kind of such regular transformations is developed in ([4] II §§ 4, 5). We first prove (5) for the simple type (2) in the form

$$x_i' = \prod_{j=0}^{j=n} (x_j)/x_i \qquad (i = 0, 1, \cdots, n). \tag{6}$$

The members of the homaloidal system contain the $n+1$ isolated F-points at the vertices of the reference $(n+1)$-edron to the multiplicity $n-1$, the $\binom{n+1}{2}$ F_1's joining any two of the vertices to the multiplicity $n-2, \cdots$; in general the $\binom{n+1}{r+1}$ F_r's joining any $r+1$ of the vertices to the multiplicity $n-r-1$ $(r = 0, \cdots, n-2)$. Let $x_0, x_1, \cdots, x_r, 0, 0, \cdots, 0$ be a generic F-point on one of these F_r's. A generic direction at this F-point on the line xy is determined by infinitesimal ϵ in

$$x_0 + \epsilon y_0, \cdots, x_r + \epsilon y_r, \quad \epsilon y_{r+1}, \cdots, \epsilon y_n. \tag{7}$$

The transform of this by (6), after factoring out ϵ^{n-r-1}, then setting $\epsilon = 0$ and removing the further factor $x_0 x_1 \cdots x_r y_{r+1} \cdots y_n$, is

$$0, 0, \cdots, 0, 1/y_{r+1}, \cdots, 1/y_n. \tag{8}$$

Thus as y runs over any S_{n-1}, not on the F-point, the point, which corresponds to the direction from F on F_r to y, runs over the F_{n-r-1}-locus determined by the $n-r$ vertices complementary to the $r+1$ which determine F_r, and this quite irrespective of the choice of F on F_r. Hence *opposite* F_r and F_{n-r-1} of the reference $(n+1)$-edron correspond as in (5). In particular the point F_0 and opposite S_{n-1} are isolated F-point and corresponding P-locus of (2).

When (2) is preceded or followed by a collineation the resulting transformation T of order n has $n+1$ isolated F-points which form one $(n+1)$-edron, and their $(n+1)$ P-loci form a dual $(n+1)$-edron whose vertices, paired with those of the other, are the isolated F-points of T^{-1}. Then (5) holds as before except that opposite F_r and F_{n-r-1} are in different $(n+1)$-edrons.

In a product of transformations T of order n the F-loci of the product must arise from F-loci of the particular factors. If a manifold M_r is converted, by applying the first k factors, into an F_r-locus of the $(k+1)$-th factor, this factor converts it into an F_{n-r-1} which is in general transformed by further factors into an M_{n-r-1}. Then M_r, M_{n-r-1} are F-loci paired as in (5). If, as factors are applied, M_{n-r-1} passes back into an M_r' then M_r, M_r' are ordinary images under the product. If however n is odd and $r =$

$n-r-1$, or $r=(n-1)/2$, a particular examination is necessary to see whether M_r, M_r' are F-loci. If they turn out to be F-loci they are paired as in (5). Hence (5) applies to all regular transformations.

We shall use from time to time two properties of the transformation (2) whose restatement for the general regular transformation *of order* n is obvious. These are:

(9) *A linear space* S_{r+1} *on* F_r *which meets the opposite* F_{n-r-1} *in* y *is transformed by* (2) *into a linear space* S'_{r+1} *on* F_r *which meets* F_{n-r-1} *in* y'. *The transformation* (2′) *between* y, y' *in* F_{n-r-1} *is of the type* (2) *and order* $n-r-1$ *with* $(n-r)$*-edron composed of the isolated* F*-points of* (2) *in* F_{n-r-1}. *The transformation between point* x *in* S_{r+1} *and point* x' *in* S'_{r+1} *is a regular transformation of order* $r+1$ *whose* $r+2$ *isolated* F*-points in* S_{r+1} *consist of the* $r+1$ F*-points of* (2) *in* F_r *and* y; *and whose* $r+2$ *isolated inverse* F*-points in* S'_{r+1} *consist of the* $r+1$ F*-points of* (2) *in* F_r *and* y'.

For, let an S_{r+1} on the F_r used in connection with (7) and (8) be defined by the equations $x_{r+1}/\lambda_{r+1}=\cdots=x_n/\lambda_n=k$. It meets the opposite F_{n-r-1} in the point $y=0,\cdots,0,\lambda_{r+1},\cdots,\lambda_n$. This S_{r+1} is transformed by (2) into the S'_{r+1} defined by $\lambda_{r+1}x_{r+1}=\cdots\lambda_n x_n=1/k$, which meets F_{n-r-1} in the point $y'=0,\cdots,0,1/\lambda_{r+1},\cdots,1/\lambda_n$, which proves the first property. From this point of view the transformation (2) in S_n has been called the *dilation* of the transformation (2′) in F_{n-r-1} ([6] pp. 256-258). When S_{r+1} is fixed by the choice of $\lambda_{r+1},\cdots,\lambda_n$, a point x on S_{r+1} has coördinates in S_n $x_0, x_1,\cdots,x_r, k\lambda_{r+1},\cdots,k\lambda_n$, and coördinates in S_{r+1}, $x_0, x_1,\cdots, x_r, k$. Its image in (2) has coördinates in S_n, $1/x_0, 1/x_1,\cdots,1/x_r, 1/k\lambda_{r+1},\cdots,1/k\lambda_n$, and coördinates in S'_{r+1}, $1/x_0, 1/x_1,\cdots,1/x_r, 1/k$, which proves the second property. Thus the transformation (2) in S_n appears as the product of subsidiary transformations of similar type in S_{n-r-1} and S_{r+1}.

3. The group G_2^{n+2} defined by a set, $P^n{}_{n+3}$ of $n+3$ points in S_n. Let $p_1, p_2,\cdots,p_{n+3}$ be the individual points of $P^n{}_{n+3}$. They are on a unique rational norm-curve, N^n, in S_n. If t is a parameter on N^n these points are determined by $t=t_1, t_2,\cdots,t_{n+3}$ respectively. Let I_{ij} $(i,j=1,\cdots,n+3)$ be the involutorial transformation of type (2) which interchanges p_i, p_j and which has isolated F-points at the remaining $n+1$ points of $P^n{}_{n+3}$. If these $n+1$ points are the reference $(n+1)$-edron and p_i, p_j are y, z respectively the equations of I_{ij} are

$$x_i'x_i=y_iz_i \qquad (i=0,\cdots,n). \tag{10}$$

The group generated by these $\binom{n+3}{2}$ involutions I_{ij} is characterized as follows:

(11) *The group generated by the involutions $I_{i_1 i_2}$ contains 2^{n+2} involutorial elements $I_{i_1 i_2 \cdots i_{2l}}$ ($l = 0, 1, \cdots, (n+2)/2$ if n is even; $l = 0, 1, \cdots, (n+3)/2$ if n is odd) where $i_1, i_2, \cdots, i_{2l}$ are any $2l$ numbers selected from $1, \cdots, n+3$. This $G_{2^{n+2}}$ is abelian of type $(1, 1, \cdots, 1)$ and the law of multiplication is*

$$I_{i_1 i_2 \cdots i_{2l}} \cdot I_{j_1 j_2 \cdots j_{2m}} = I_{j_1 j_2 \cdots j_{2m}} \cdot I_{i_1 i_2 \cdots i_{2l}} = I_{k_1 k_2 \cdots k_{2n}},$$

where $k_1, \cdots, k_{2n}$ are the numbers found in one but not in both of the sets $i_1 \cdots i_{2l}$ and $j_1 \cdots j_{2m}$. The order of the subscripts in $I_{i_1 \cdots i_{2l}}$ is not material.

Thus $I_{12}I_{13} = I_{13}I_{12} = I_{23}$, $I_{12}I_{34} = I_{13}I_{24} = I_{1234}$, etc. Given this law of multiplication, the elements are necessarily involutorial and the number of elements is $\binom{n+3}{0} + \binom{n+3}{2} + \cdots = 2^{n+2}$. In order to prove the law it is necessary to be able to define the individual elements. This may be done most conveniently by introducing the *principal* loci or P-loci of the set of points $P^n{}_{n+3}$. The P-loci of $P^n{}_{n+3}$ are the P-loci which correspond to directions about the isolated F-points $p_1, \cdots, p_{n+3}$ under any regular transformation T whose isolated F-points are all included in the set $P^n{}_{n+3}$. Since T is generated by elements of type $I_{i_1 i_2}$, all P-loci of the set $P^n{}_{n+3}$ are obtained by applying these generating elements to the sets of directions at points p and by repeating the application to the new P-loci as they arise. The P-loci are all comprised in the aggregate,

$$\pi_{i_1 \cdots i_{2k+1}} = (i_1{}^{k-1} \cdots i^{k-1}_{2k+1}\, i^{k}_{2k+2} \cdots i^{k}_{n+3})^{k}_{n-1} \qquad (1 \leqq 2k+1 \leqq n+3), \tag{12}$$

where the symbol on the right indicates a manifold of dimension $n-1$, and order k, with multiple points of order $k-1$ at $p_{i_1}, \cdots, p_{i_{2k+1}}$ and of order k at $p_{i_{2k+2}}, \cdots, p_{i_{n+3}}$. Included in the aggregate is

$$\pi_{i_1} = (i_1{}^{-1})^{0}_{n-1} \tag{13}$$

which represents the set of directions at p_i. That this P-locus, $\pi_{i_1 \cdots i_{2k+1}}$, exists and is uniquely determined by the given order and multiplicities will be clear in the sequel where it appears that it can be transformed into the unique linear space on n given points. The rule for transforming these P-loci by the generating involutions is as follows:

(14) *The P-locus $\pi_{i_1 \cdots i_{2k+1}}$ is transformed by the involution $I_{j_1 j_2}$ into the P-locus $\pi_{i_1 \cdots i_{2k+1} j_1 j_2}$ provided that in the aggregate of subscripts, $i_1 \cdots i_{2k+1} j_1 j_2$, like subscripts cancel.*

Because of the symmetry of π in (12) in the two complementary sets of indices it is sufficient to verify (14) for the three cases $I_{i_{2k}i_{2k+1}}$, $I_{i_{2k+1}i_{2k+2}}$ and $I_{i_{2k+2}i_{2k+3}}$. Under $I_{i_{2k}i_{2k+1}}$, π becomes

$$\left(i_1{}^{(n-1)k} \cdots i_{2k-1}^{(n-1)k}\ i_{2k}^{k-1}\ i_{2k+1}^{k-1}\ i_{2k+2}^{(n-1)k} \cdots i_{n+3}^{(n-1)k}\right)^{nk},$$

from which, due to the given multiplicities of π at the F-points of $I_{i_{2k}i_{2k+1}}$, the linear spaces $\pi_{i_j i_{2k}i_{2k+1}}$ for $j = 1, \cdots, 2k-1$ must each factor $k-1$ times, and the linear spaces $\pi_{i_j i_{2k}i_{2k+1}}$ for $j = 2k+2, \cdots, n+3$ must each factor k times. The order of the residual factor is $k-1 = nk - (k-1)(2k-1) - k(n+3-2k-1)$; The multiplicity of the factor at $p_{i_1}, \cdots, p_{i_{2k-1}}$ is $(n-1)k - (k-1)(2k-2) - k(n-2k+2) = k-2$; and the multiplicity at $p_{i_{2k+2}}, \cdots, p_{i_{n+3}}$ is $(n-1)k - (k-1)(2k-1) - k(n-2k+1) = k-1$. Hence the image of $\pi_{i_1 \ldots i_{2k+1}}$ is $\pi_{i_1 \ldots i_{2k-1}}$ or $\pi_{i_1 \ldots i_{2k+1}i_{2k}i_{2k+1}}$ according to the rule (14), provided there is a unique manifold of this type. The second and third cases can be verified similarly if in the second case account is taken also of the interchange of $p_{i_{2k+1}}$ and $p_{i_{2k+2}}$. Thus by a sequence of properly chosen involutions $I_{j_1 j_2}$, π can be transformed into $\pi_{i_1 i_2 i_3}$ which, as a linear space on $p_{i_4}, \cdots, p_{i_{n+3}}$, exists and is unique. Hence π exists and is unique.

A product of involutions $I_{j_1 j_2}$, or general member of the group (11), is defined when the P-loci corresponding to its isolated F-points, $p_1, \cdots, p_{n+3}$, are given; i. e., if the images of the P-loci π_{i_1} are given. Under the product $I_{12}I_{13}$ according to the rule of (14), π_1 becomes π_2 and then π_{123}; π_2 becomes π_1 and then π_3; π_3 becomes π_{123} and then π_2; while π_i $(i = 4, \cdots, n+3)$ becomes π_{i12} and then π_{i23}. Hence $I_{12}I_{13} = I_{13}I_{12} = I_{23}$. Under the product $I_{12}I_{34}$, π_i becomes π_{i12} and then π_{i1234}. This result, symmetric in the two sets of indices, $i = 1, 2, 3, 4$ and $i = 5, \cdots, n+3$, shows that $I_{12}I_{34} = I_{34}I_{12} = I_{13}I_{42} = \cdots = I_{1234}$. The continuation to complete the proof of (11) is obvious.

(15) *The general element $I_{i_1 i_2 \ldots\ 2l}$ of $G_2{}^{n+2}$ converts linear spaces into members of the homaloidal system*

$$\left(i_1{}^{(l-1)(n-1)} \cdots i_{2l}^{(l-1)(n-1)}\ i_{2l+1}^{l(n-1)} \cdots i_{n+3}^{l(n-1)}\right)_{n-1}^{1+l(n-1)}.$$

The P-locus of the isolated F-point p_{i_1} is $\pi_{i_2 \ldots i_{2l}}$; of the isolated F-point $p_{i_{2l+1}}$ is $\pi_{i_1 \ldots i_{2l}i_{2l+1}}$.

For, the theorem is true when $l = 0$ (the identity), and when $l = 1$ (a generating involution). Assuming that it is true for values of l up to and including $k-1$, we find by actual multiplication that $I_{i_1 \ldots i_{2k-2}} \cdot I_{i_{2k-1}i_{2k}}$ has a homaloidal system of the form (15) for $l = k$. If then $I_{i_1 \ldots i_{2l}}$ is

expressed as a product $I_{i_1 i_2} \cdot I_{i_3 i_4} \cdot \ldots \cdot I_{i_{2l-1} i_{2l}}$, the effect upon the sets of directions π_j is obtained from the rule (14).

Specific equations for the involution $I_{i_1 \ldots i_{2l}}$ in terms of the P-loci can be given. If we choose as reference points, $p_{i_3}, \cdots, p_{i_{2l}}, p_{i_{2l+1}}, \cdots, p_{i_{n+3}}$, the reference spaces opposite these points are respectively $\pi_{i_1 i_2 i_3}, \cdots, \pi_{i_1 i_2 i_{2l}}$, $\pi_{i_1 i_2 i_{2l+1}}, \cdots, \pi_{i_1 i_2 i_{n+3}}$. Then the image of $\pi_{i_1 i_2 i_3}$ under $I_{i_1 i_2 \ldots i_{2l}}$ is composed of the proper image of $\pi_{i_1 i_2 i_3}$ which, according to (14), is $\pi_{i_4 \ldots i_{2l}}$, and of the P-loci of the n isolated F-points on $\pi_{i_1 i_2 i_3}$. If the $(n+1)$ P-loci of the reference points are divided out, the equations of $I_{i_1 \ldots i_{2l}}$ take the form

$$
\begin{aligned}
&\pi'_{i_1 i_2 i_3} = k_{i_3} \frac{\pi_{i_4 \ldots i_{2l}}}{\pi_{i_1 i_2 i_4 \ldots i_{2l}}}; \qquad \pi'_{i_1 i_2 i_{2l+1}} = k_{i_{2l+1}} \frac{\pi_{i_3 \ldots i_{2l} i_{2l+1}}}{\pi_{i_1 i_2 i_3 \ldots i_{2l} i_{2l+1}}}; \\
(16) \quad &\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots \\
&\pi'_{i_1 i_2 i_{2l}} = k_{i_{2l}} \frac{\pi_{i_3 \ldots i_{2l-1}}}{\pi_{i_1 i_2 i_3 \ldots i_{2l-1}}}; \qquad \pi'_{i_2 i_2 i_{n+3}} = k_{i_{n+3}} \frac{\pi_{i_3 \ldots i_{2l} i_{n+3}}}{\pi_{i_1 i_2 i_3 \ldots i_{2l} i_{n+3}}}.
\end{aligned}
$$

In order to obtain specific equations for the P-loci themselves we must observe that their geometric definitions as given in (12) in terms of the multiplicities at the points of $P^n{}_{n+3}$ are too precise. For example, $\pi_{i_1 \ldots i_{2k+1}}$ of order k with k-fold points at $p_{i_{2k+2}}, \cdots, p_{i_{n+3}}$ has k-fold points at all the points of the linear S_{n+1-2k} on these given $(n+2-2k)$ k-fold points. Also if N^n is the unique rational norm curve of order n on $P^n{}_{n+3}$, then π has $(k-1)$-fold points not merely at $p_{i_1}, \cdots, p_{i_{2k+1}}$ but also at all points on N^n. This is a consequence of the following theorem in which the points $p_{i_1}, \cdots, p_{i_{2k+1}}$ do not appear explicitly.

(17) *The P-locus $\pi_{i_1 \ldots i_{2k+1}}$ $[0 < k \gtrless (n+2)/2]$ is the locus of the ∞^{k-1} linear spaces S_{n-k} which are $(n+1-k)$-secant to N^n, and on the fixed group of $n+2-2k$ points, $p_{i_{2k+2}}, \cdots, p_{i_{n+3}}$, which determines the k-fold locus S_{n+1-2k} of the P-locus.*

For, the spaces S_{n-k}, projected from S_{n+1-2k}, become spaces S_{k-2} in S_{2k-2} which are $(k-1)$-secant to the unique N^{2k-2} on the projections of $p_{i_1}, \cdots, p_{i_{2k+1}}$. According to the lemma (18) these spaces S_{k-2} make up the manifold of order k with $(k-1)$-fold points at the projections of $p_{i_1}, \cdots, p_{i_{2k+1}}$ and the proof of (17) is complete.

(18) *In the space S_{2k-2} the manifold of dimension $2k-3$ and order k with $(k-1)$-fold points at $p_1, \cdots, p_{2k+1}$ has $(k-1)$-fold points all along the N^{2k-2} on the P^{2k-2}_{2k+1}, and is the locus of the S_{k-2}'s which are $(k-1)$-secant to N^{2k-2}. If the coefficients of a binary $(2k-2)$-ic are taken as coördinates in S_{2k-2} and perfect powers represent points on N^{2k-2}, the equation of this manifold is given by the vanishing of the canonizant of the binary $(2k-2)$-ic.*

Let the binary $(2k-2)$-ic be represented symbolically by

$$(19) \qquad (a_1t)^{2k-2} = (a_2t)^{2k-2} = (a_3t)^{2k-2} = \cdots.$$

We use only one type of symbolic factor which, for two cogredient symbols, for two cogredient variables, and for contragredient symbol and variable, is to indicate respectively

$$(20) \qquad (a_ia_j) = a_{i0}a_{j1} - a_{i1}a_{j0}; \qquad (t_it_j) = t_{i0}t_{j1} - t_{i1}t_{j0};$$
$$(a_it_j) = a_{i0}t_{j0} + a_{i1}t_{j1}.$$

The canonizant of the $(2k-2)$-ic is, symbolically,

$$(21) \qquad (a_1a_2)^2(a_1a_3)^2(a_2a_3)^2 \cdots (a_{k-1}a_k)^2;$$

and nonsymbolically,

$$(22) \qquad \begin{vmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ a_1 & a_2 & \cdots & a_k \\ \cdot & \cdot & \cdots & \cdot \\ a_{k-1} & a_k & \cdots & a_{2k-2} \end{vmatrix}.$$

The two-row minors of the determinant (22) of order k are of the form $a_ia_{i+j+k} - a_{i+j}a_{i+k}$, and vanish when the $(2k-2)$-ic is a perfect power. Hence the manifold (22) of order k has $(k-1)$-fold points on N^{2k-2}. The manifold of order k with $(k-1)$-fold points at $p_1, \cdots, p_{2k+1}$ is a P-locus in S_{2k-2} and is uniquely determined. It must therefore coincide with (22). If the canonizant vanishes, the $(2k-2)$-ic can be represented as a sum of $(k-1)$ perfect powers, and the point in S_{2k-2} represented by the $(2k-2)$-ic is in the S_{k-2} determined by the corresponding $k-1$ points on N^{2k-2}.

(23) *In the coördinate system in S_n with reference to N^n described in* (18)-(20), *the equation of the P-locus $\pi_{i_1 \ldots i_{2k+1}}$ is*

$$(a_1a_2)^2(a_1a_3)^2(a_2a_3)^2 \cdots (a_{k-1}a_k)^2(a_1t_{i_{2k+2}}) \cdots (a_1t_{i_{n+3}}) \cdots (a_kt_{i_{2k+2}}) \cdots (a_kt_{i_{n+3}}) = 0,$$

where $t_{i_{2k+2}}, \cdots, t_{i_{n+3}}$ are the parameters on N^n of $p_{i_{2k+2}}, \cdots, p_{i_{n+3}}$.

For, if, as in (17), a point of π is on an S_{n-k} which is $(n+1-k)$-secant to N^n, the corresponding binary n-ic can be expressed as a sum of $(n+1-k)$ perfect n-th powers of which $n+2-2k$ are the powers, $(t_{i_{2k+2}}t)^n, \cdots, (t_{i_{n+3}}t)^n$. The polar of $(t_{i_{2k+2}}t) \cdots (t_{i_{n+3}}t)$ as to this n-ic is a $(2k-2)$-ic which can be expressed as a sum of perfect $(2k-2)$-th powers of the remaining $k-1$ linear forms, whence the catalecticant of this polar vanishes.

4. Conjugate sets of F-loci of G_2^{n+2} of the j-th kind. The isolated

F-points and the P-loci of P^n_{n+3} may *be classed together as the F-loci of the first kind of* G_2^{n+2}. Of course a P-locus cannot be an actual F-locus of any element of G_2^{n+2}, but it is conjugate to the isolated F-points under G_2^{n+2} and for that reason may be classed with them.

We shall define the *k-th F-locus of* G_2^{n+2} *of the j-th kind,* $\pi^{(j)}{}_{i_1 i_2 \dots i_{2k-j+2}}$ $[1 \gtrsim j \gtrsim (n+1)/2]$ to be, when $j \gtrsim k \gtrsim (n+j+1)/2$, the locus of dimension $n-j$ which is described by the ∞^{k-j} S_{n-k}'s on $p_{i_{2k-j+3}}, \cdots, p_{i_{n+3}}$, and on $k-j$ variable points of N^n; and when $(j-2)/2 \gtrsim k < j$, the locus of dimension $j-1$ which is described by the ∞^{k-j-1} S_k's on $p_{i_1}, \cdots, p_{i_{2k-j+2}}$, and on $j-k-1$ variable points of N^n. Of especial simplicity are the cases $k=j$ and $k=j-1$ for which the F-loci are linear spaces of respectively dimension $n-j$ on $p_{i_{j+3}}, \cdots, p_{i_{n+3}}$, and dimension $j-1$ on $p_{i_1}, \cdots, p_{i_j}$. These are corresponding F-loci under the element $I_{i_{j+1} i_{j+2}}$ of G_2^{n+2} (cf. **2**). Of interest also are the end cases in which the points of P^n_{n+3} occur symmetrically. These occur, for $k \gtrsim j$, when $2k-j = n+1$ and the F-locus consists of the ∞^{k-j} S_{n-k}'s which are $(k-j)$-secant to N^n, and, for $k < j$, when $2k = j-2$ and the F-locus consists of the $\infty^{j/2}$ $S_{(j-2)/2}$'s which are $(j/2)$-secant to N^n. Thus for $j=2$ we have the norm-curve N^n itself, i. e., $\pi^{(2)} = N^n$.

These F-loci are all of the same character as the P-loci, i. e. they are loci of multi-secant spaces of N^n. If we use the theorem of Castelnuovo (cf. [3] p. 155) which states that the spaces S_{m-1}, m-secant to N^n, make up an $M^{\binom{n-m+1}{m}}_{2m-1}$ on which N^n is a curve of multiplicity $\binom{n-m}{m-1}$, it is easy to show that

(24) *The F-locus* $\pi^{(j)}{}_{i_1 \dots i_{2k-j+2}}$ $(k \gtrsim j)$, *has the order* $\binom{k}{j}$, *the multiplicity* $\binom{k}{j}$ *on the* S_{n-2k+j} *defined by* $p_{i_{2k-j+3}}, \cdots, p_{i_{n+3}}$, *and the multiplicity* $\binom{k-1}{j}$ *along* N^n. *The F-locus,* $\pi^{(j)}{}_{i_1 \dots i_{2k-j+2}}$ $(k < j)$, *has the order* $\binom{n-k}{j-k-1}$, *the multiplicity* $\binom{n-k}{j-k-1}$ *on the* S_{2k-j+1} *defined by* $p_{i_1}, \cdots, p_{i_{2k-j+2}}$, *and the multiplicity* $\binom{n-k-1}{j-k-2}$ *along* N^n.

For, in the first case, the F-locus is made up of the ∞^{k-j} S_{n-k}'s on S_{n-2k+j} which are further $(k-j)$-secant to N^n. Projected from the S_{n-2k+j}, they become the ∞^{k-j} S_{k-j-1}'s which are $(k-j)$-secant to the projected N^{2k-j-1}. The latter locus has the order $\binom{k}{k-j}$ and contains N^{2k-j-1} with multiplicity $\binom{k-1}{k-j-1}$, which determine the order and multiplicities of the original locus. In the second case, the F-locus is made up of the ∞^{j-k-1} S_k's on S_{2k-j+1} which are further $(j-k-1)$-secant to N^n. Projected from S_{2k-j+1}, they become the ∞^{j-k-1} S_{j-k-2}'s which are $(j-k-1)$-secant to the

projected $N^{n-2k+j-2}$. The latter locus has the order $\binom{n-k}{j-k-1}$ and contains $N^{n-2k+j-2}$ to the multiplicity $\binom{n-k-1}{j-k-2}$.

Equations, analogous to those of (23), of linear systems of manifolds of dimension $n-1$, whose base is a particular F-locus, can be given. We observe that if a point of S_n lies in an S_{r+s-1} which is $(r+s)$-secant to N^n at $r+s$ points of which r are fixed at $p_1, \cdots, p_r$, then the binary n-ic, $(a_1t)^n$, whose coefficients determine the point, can be expressed as a sum of $r+s$ perfect n-th powers, r of which are $(t_1t)^n, \cdots, (t_rt)^n$. The polar $(n-r)$-ic, $(a_1t_1)\cdots(a_1t_r)(a_1t)^{n-r}$, can therefore be expressed as a sum of s perfect $(n-r)$-th powers. If then $(\alpha t)^{n-r-2s}$ is an arbitrary binary form of the order indicated, the polar $2s$-ic, $(a_1\alpha)^{n-r-2s}(a_1t_1)\cdots(a_1t_r)(a_1t)^{2s}$, can be expressed as a sum of s perfect $2s$-th powers, and its catalecticant vanishes for all values of the coefficients α. Naturally we must have $r+2s \leqq n$. The locus of such S_{r+s-1}'s is then defined by the linear system of manifolds of order $s+1$ [i. e., degree $s+1$ in the coefficients of $(a_1t)^n$],

$$(25)\qquad \begin{aligned}(a_1a_2)^2(a_1a_3)^2(a_2a_3)^2\cdots(a_sa_{s+1})^2(a_1\alpha_1)^{n-r-2s}\cdots(a_{s+1}\alpha_{s+1})^{n-r-2s}\cdot\\ (a_1t_1)\cdots(a_1t_r)\cdots(a_{s+1}t_1)\cdots(a_{s+1}t_r)=0,\end{aligned}$$

i. e., the coefficients of the combinations of degree $s+1$ of the coefficients of $(\alpha t)^{n-r-2s}$ are themselves members of a linear system which has for its base the locus of the $(r+s)$-secant spaces. On applying this to the F-loci (24) we find that

(26) *For* $r=n-2k+j+1$, $s=k-j$, *and* $t_1, \cdots, t_r = t_{i_{2k-j+2}}, \cdots, t_{i_{n+3}}$ *the F-locus,* $\pi^{(j)}{}_{i_1\ldots i_{2k-j+2}}$ $(k \geqq j)$, *is defined by the linear system* (25) *in which the coefficients of* $(\alpha t)^{n-r-2s}=(\alpha t)^{j-1}$ *are the parameters; the F-locus* $\pi^{(j)}{}_{i_1\ldots i_{2k-j+2}}$ $(k<j)$, *is similarly defined when* $r=2k-j+2$, $s=j-k-1$, $t_1, \cdots, t_r = t_{i_1}, \cdots, t_{i_{2k-j+2}}$, *and* $(\alpha t)^{n-r-2s}=(\alpha t)^{n-j}$.

The number of F-loci of the j-th kind is either the number of even, or the number of odd, combinations of $n+3$ things; and thus is always 2^{n+2}, except when n is odd and $j=(n+1)/2$. In this case the 2^{n+2} types coalesce in pairs into 2^{n+1} types (see table at the end of this section). That they form a conjugate set under $G_2{}^{n+2}$ is a consequence of the theorem:

(27) *The F-locus,* $\pi^{(j)}{}_{i_1\ldots i_{2k-j+2}}$, *is transformed by* $I_{j_1\ldots j_{2l}}$ *into the F-locus,* $\pi^{(j)}{}_{i_1\ldots i_{2k-j+2}j_1\ldots j_{2l}}$ *where in the result like subscripts are to be deleted.*

It is sufficient to prove that the behavior of the F-loci is as stated in (27) for the generating involutions alone. We also have noted already that the linear F-loci for $k=j-1$ and $k=j$ are interchanged, i. e., that $\pi^{(j)}{}_{i_1\ldots i_j}$

is transformed by $I_{i_{j+1}i_{j+2}}$ into $\pi^{(j)}{}_{i_1 \dots i_{j+2}}$ and vice versa. We have to prove further only that, for $k \geqq j$, (a) $\pi^{(j)}{}_{i_1 \dots i_{2k-j+2}}$ is transformed by $I_{i_{2k-j+2}i_{2k-j+3}}$ into $\pi^{(j)}{}_{i_1 \dots i_{2k-j+1}i_{2k-j+3}}$; and that (b) $\pi^{(j)}{}_{i_1 \dots i_{2k-j+3}}$ is transformed by $I_{i_{2k-j+3}i_{2k-j+4}}$ into $\pi^{(j)}{}_{i_1 \dots i_{2k-j+4}}$; as well as that, for $k < j$, (c) $\pi^{(j)}{}_{i_1 \dots i_{2k-j+3}}$ is transformed by $I_{i_{2k-j+2}i_{2k-j+3}}$ into $\pi^{(j)}{}_{i_1 \dots i_{2k-j+1}i_{2k-j+3}}$; and that (d) $\pi^{(j)}{}_{i_1 \dots i_{2k-j+3}}$ is transformed by $I_{i_{2k-j+1}i_{2k-j+2}}$ into $\pi^{(j)}{}_{i_1 \dots i_{2k-j}}$.

In the case (a) both the original and transformed F-loci are made up of linear spaces on the $S_{n-2k+j-1}$ determined by the $n-2k+j$ points $p_{i_{2k-j+2}}, \dots, p_{i_{n+3}}$. According to (9) it is sufficient to prove the theorem for the projections of the F-loci from $S_{n-2k+j-1}$. Thus the proof for case (a) is reduced to the proof for case (a′): in S_{2k-j} to prove that the ∞^{k-j} S_{k-j}'s on $p_{i_{2k-j+3}}$ and $k-j$ variable points of N^{2k-j} is transformed by $I_{i_{2k-j+2}i_{2k-j+3}}$ into the ∞^{k-j} S_{k-j}'s on $p_{i_{2k-j+2}}$ and $k-j$ variable points of N^{2k-j}. Similarly case (b) is reduced to case (b′): in S_{2k-j+1} to prove that the ∞^{k-j} S_{k-j+1}'s on $p_{i_{2k-j+3}}, p_{i_{2k-j+4}}$ and $k-j$ variable points of N^{2k-j+1} is transformed by $I_{i_{2k-j+3}i_{2k-j+4}}$ into the ∞^{k-j+1} S_{k-j}'s on $k-j+1$ variable points of N^{2k-j+1}. Similar projections for cases (c) and (d) convert them into cases (c′) and (d′) which arise from cases (a′) and (b′) by setting $k = n-k$ and $j = n-j+1$. If now in case (a′) we set $2k-j = n$ and $k-j = r$, and in case (b′) we set $2k-j+1 = n$ and $k-j+1 = r$ they read as follows: Case (a″): in S_n to prove that the ∞^r S_r's on p_{n+3} and r variable points of N^n is transformed by $I_{n+2,n+3}$ into the ∞^r S_r's on p_{n+2} and r variable points of N^n; case (b″): in S_n to prove that the ∞^{r-1} S_r's on p_{n+2}, p_{n+3} and $r-1$ variable points of N^n is transformed by $I_{n+2,n+3}$ into the ∞^r S_{r-1}'s on r variable points of N^n. Thus the proof of (27) for the entire set of values of k is made to depend on the proof for the extreme value of k only and for the extreme generating involution $I_{n+2,n+3}$. The proof for case (a″) can be carried through precisely like the proof for case (b″) which we proceed to give.

In case (b″) let M_{2r-1} be the manifold described by the ∞^r r-secant S_{r-1}'s of N^n, and M'_{2r-1} that described by the ∞^{r-1} $(r+1)$-secant S_r's of N^n on p_{n+2}, p_{n+3}. The involution $I_{n+2,n+3}$ has isolated F-points at $p_1, \dots, p_{n+1}$. Let g be any set of $n-2r$ of these $n+1$ F-points, say $p_1, \dots, p_{n-2r}$, in an S_{n-2r-1}. If this S_{n-2r-1} be joined to each point of M_{2r-1}, and to each point of M'_{2r-1}, by an S_{n-2r}, the loci which thus arise are the two P-loci of, respectively, the ∞^r S_{n-r-1}'s on $p_1, \dots, p_{n-2r}$ and r variable points of N^n, and the ∞^{r-1} S_{n-r}'s on $p_1, \dots, p_{n-2r}, p_{n+2}, p_{n+3}$ and $r-1$ variable points on N^n, i. e., the P-loci $\pi_{n-2r+1, \dots, n+3}$ and $\pi_{n-2r+1, \dots, n+1}$. According to (14) these two P-loci are interchanged by $I_{n+2,n+3}$. If we prove that M_{2r-1} is the complete intersection of the various P-loci obtained from the $\binom{n+1}{n-2r}$ choices

of the set g, and that M'_{2r-1} is a similar complete intersection, then these two complete intersections are interchanged by $I_{n+2,n+3}$. We examine the case of M_{2r-1} first. Let x be any point of S_n on all the P-loci determined by M_{2r-1} and the choice of the set g, and let x be determined by the coefficients of the binary n-ic, $(at)^n$. Since x is on the particular P-locus given above, $(at)^n$ can be expressed in terms of the n-th powers

$$(t_1t)^n, \cdots, (t_{n-2r-1}t)^n, \ (t_{n-2r}t)^n, \ (s_1t)^n, \cdots, (s_rt)^n.$$

Thus the polar of $(t_1t)\cdots(t_{n-2r-1}t)$ as to $(at)^n$ is of order $2r+1$, and either t_{n-2r} is a root of the canonizant (of order $r+1$) of this polar, or the canonizant vanishes identically. But, as g varies, t_{n-2r} can be replaced by $t_{n-2r+1}, \cdots, t_{n+1}$, and the canonizant with these $2r+2$ distinct roots must vanish identically. But $(t_1t)\cdots(t_{n-2r-1}t)$ is any one of $\binom{n+1}{n-2r-1}$ products which may be formed from $(t_1t), \cdots, (t_{n+1}t)$ and any form of order $n-2r-1$ can be expressed linearly in terms of such products. If every polar of $(at)^n$ of order $2r+1$ can be expressed in terms of less than $r+1$ $(2r+1)$-th powers, then $(at)^n$ can be expressed in terms of less than $r+1$ n-th powers, and x is on M_{2r-1}. The proof that M'_{2r-1} is a complete intersection can be carried through in the same fashion and the proof for case (b″) is complete. The analogous proof for case (a″) completes the proof of (27).

The following tabular description of these F-loci in S_2, S_3, S_4, S_5 will be useful both by way of illustration and in connection with the more detailed discussion later of the manifolds W_3 in S_5 and W_3' in S_4:

j	k	*No.*	*Notation*	*Description*
		For P_5^2 in S_2:		
1	0	5	π_1	$^*p_1$;
1	1	10	π_{123}	$S_1(45)^1$;
1	2	1	π_{12345}	$M_1(1\cdots5)^2$.
		For P_6^3 in S_3:		
1	0	6	π_1	$^*p_1$;
1	1	20	π_{123}	$S_2(456)^1$;
1	2	6	$\pi_{1\ldots5}$	$M_2(1\cdots56^2)^2 = S_1$'s on p_6 to points of N^3;
2	0	1	$\pi^{(2)}$	N^3;
2	1	15	$\pi^{(2)}{}_{12}$	$S_1(12)^1$;
2	2	15	$\pi^{(2)}{}_{1\ldots4}$	$S_1(56)^1$;
2	3	1	$\pi^{(2)}{}_{1\ldots6}$	N^3;
		For P_7^4 in S_4:		
1	0	7	π_1	$^*p_1$;
1	1	35	π_{123}	$S_3(4567)^1$;
1	2	21	$\pi_{\ldots5}$	$M_3(1\cdots56^27^2)^2 = S_2$'s on p_6p_7 to points of N^4;

1	3	1	$\pi_{1\dots7}$	$M_3(1^2\cdots7^2)^3 =$ bisecants of N^4;
2	0	1	$\pi^{(2)}$	N^4;
2	1	21	$\pi^{(2)}{}_{12}$	$S_1(12)^1$;
2	2	35	$\pi^{(2)}{}_{1\dots4}$	$S_2(567)^1$;
2	3	7	$\pi^{(2)}{}_{1\dots6}$	$M_2(1\cdots67^3)^3 = S_1$'s on p_7 to points of N^4.

For P_8^5 in S_5:

1	0	8	π_1	$^*p_1$;
1	1	56	π_{123}	$S_4(4\cdots8)^1$;
1	2	56	$\pi_{1\dots5}$	$M_4(1\cdots56^27^28^2)^2 = S_3$'s on $p_6p_7p_8$ to points of N^5;
1	3	8	$\pi_{1\dots7}$	$M_4(1^2\cdots7^28^3)^3 = S_2$'s on p_8 to two points of N^5;
2	0	1	$\pi^{(2)}$	N^5;
2	1	28	$\pi^{(2)}{}_{12}$	$S_1(12)^1$;
2	2	70	$\pi^{(2)}{}_{1\dots4}$	$S_3(5678)^1$;
2	3	28	$\pi^{(2)}{}_{1\dots6}$	$M_3(1\cdots67^38^3)^3 = S_2$'s on p_7p_8 to points of N^5;
2	4	1	$\pi^{(2)}{}_{1\dots8}$	$M_3(1^3\cdots8^3)^6 =$ bisecants of N^5;
3	1	8	$\pi^{(3)}{}_1$	$M_2(1^42\cdots8)^4 = S_1$'s on p_1 to points of N^5;
3	2	56	$\pi^{(3)}{}_{123}$	$S_2(123)^1$;
3	3	56	$\pi^{(3)}{}_{1\dots5}$	$S_2(678)^1$;
3	4	8	$\pi^{(3)}{}_{1\dots7}$	$M_2(1\cdots78^4)^4 = S_1$'s on p_8 to points of N^5.

The doubling up of the types for $j=2$ in S_3 and for $j=3$ in S_5 is evident. This for odd spaces is an immediate consequence of the definition of the F-loci.

5. The symmetrical element in $G_{2^{n+2}}$ when n is odd. If n is odd, say

$$n+3=2p+2, \tag{28}$$

the group contains the symmetrical element $I_{1,2,\dots,2p+2}=I$ whose homaloidal system (cf. 15),

$$(1^{2p(p-1)},\cdots,2p+2^{2p(p-1)})^{2p^2-1}_{2p-2}, \tag{29}$$

contains all the points of P^{2p-1}_{2p+2} symmetrically. Under I the P-loci of the set of points are paired as follows:

$$\pi_{i_1\dots i_{2p+1}},\quad \pi_{i_{2p+2}\dots i_{2p+2}}, \tag{3}$$

Since I is invariant in the abelian group these pairs are permuted by $G_{2^{n+2}}$ as entities.

According to the definition (12) of the P-loci, and according to (17), (18), the product of the members of a pair (30) is a spread of order p with $(p-1)$-fold points at P^{2p-1}_{2p+2} and with the $(p-2)$-fold curve N^{2p-1}. The pairs for which $k=0$ are somewhat exceptional. The member π_{i_1}, the directions at p_{i_1}, is a spread of order 0 with multiplicity at p_{i_1} of virtual order -1. The complementary member, $\pi_{i_2\dots i_{2p+2}}$, has at p_{i_1} a point of

order p, and has N^{2p-1} as a $(p-1)$-fold curve (cf. 18). Thus in the linear system Σ of spreads of order p with $(p-1)$-fold points P^{2p-1}_{2p+2}, the appearance of p_{i_1} to multiplicity p in $\pi_{i_2 \dots i_{2p+2}}$ is the algebraic equivalent of the appearance of the complementary member π_{i_1}.

According to (18) the linear system Σ will contain a one-parameter system of spreads with $(p-1)$-fold points all along the curve N^{2p-1}. For, if t is any point of N^{2p-1}, the locus of the ∞^{p-1} S_{p-1}'s which are p-secant to N^{2p-1} and which pass through t has the order p and the $(p-1)$-fold curve N^{2p-1}. The equation of this one-parameter system with parameter t is, in the notation of (19), $\cdots$, (23),

$$(31) \qquad (a_1a_2)^2(a_1a_3)^2(a_2a_3)^2 \cdots (a_{p-1}a_p)^2(a_1t)(a_2t) \cdots (a_pt) = 0.$$

When t takes the particular values $t_1, t_2, \cdots, t_{2p+2}$, the products (30) for $k = 0$ and $i_1 = 1, \cdots, 2p+2$ are obtained.

The system Σ is an invariant linear system under the generators $I_{i_1 i_2}$ and therefore an invariant linear system under $G_{2^{n+2}}$. The one-parameter system (31) is characterized by the fact that it contains the F-locus of the second kind, N^{2p-1}, to a multiplicity one greater than the normal multiplicity for spreads of Σ. But the system (31) also contains the F-locus of the second kind, $\pi^{(2)}{}_{1,2,\dots,2p+2}$, the ∞^{p-1} S_{p-2}'s which are $(p-1)$-secant to N^{2p-1}, simply, i. e., to a multiplicity one greater than the normal multiplicity for spreads of Σ. Hence

(31a) *The linear system Σ of spreads of order p with $(p-1)$-fold points at P^{2p-1}_{2p+2} invariant under $G_{2^{n+2}}$ contains 2^{2p} members which break up into the members of a pair* (30). *The F-loci of the second kind of $G_{2^{n+2}}$ divide into 2^{2p} pairs, each a pair of conjugates under I,*

$$\pi^{(2)}{}_{i_1 \dots i_{2k}}, \quad \pi^{(2)}{}_{i_{2k+1} \dots i_{2p+2}}.$$

The members of Σ which contain one of these pairs of F-loci of the second kind to a multiplicity one greater than the normal multiplicity of Σ for the pair constitute a linear system of dimension p which contains a non-linear system of dimension one and degree p. Each of these non-linear systems contains $2p+2$ of the pairs (30) *whose parameters are projective to the fundamental $(2p+2)$-ic. Under I each member of Σ is invariant. Under the factor group of $G_{2^{n+2}}$ with respect to I, a $g_{2^{2p}}$, the 2^{2p} pairs* (30), *and the 2^{2p} non-linear systems, are permuted regularly.*

We prove in the next section that I has a locus, W_p, of fixed points. Anticipating this result for the moment we find the loci of fixed points of the various involutorial elements of $G_{2^{n+2}}$ for n odd or even.

6. The loci of fixed points of elements of G_2^{n+2}. We consider first the rather well-known cases, $n=2$, $n=3$. For $n=2$, the G_2^4 in S_2 has 10 elements of type I_{ij} and five of type I_{ijkl} $(i,\cdots,l=1,\cdots,5)$. The F-loci are all of P-type, the five sets of directions π_i, the ten lines $\pi_{ijk}=p_lp_m$, and the conic π_{12345}. The quadratic involution I_{ij} is non-perspective since p_i, p_j are not a line with any F-point. It has therefore four fixed points. The cubic involution I_{ijkl} has, however, a cubic locus of fixed points. For, under $I_{ijkl}=I_{ij}I_{kl}$ the line p_mp_i is transformed into p_mp_i whence every line of the pencil on p_m is invariant. The line p_ip_j is transformed into p_kp_l, whence the three degenerate conics, and therefore all conics, on $p_i,\cdots,p_l$ are invariant. The conics of this pencil are projected, each into itself, from the center p_m; and the locus of fixed points is generated by the pencil of conics, and the pencil of polar lines of p_m. Thus the locus is a cubic curve on P_5^2, tangent at p_m to π_{12345}, and on the diagonal triangle of $p_i,\cdots,p_l$.

For $n=3$ and G_2^5 with involutions I_{ij}, I_{ijkl}, $I_{1\ldots6}$, the well known cubic involution I_{ij} has 8 fixed points which divide into two tetrahedra which, with the F-tetrahedron, make up a desmic system. The involution I_{1234} has a curve of fixed points. For, under $I_{1234}=I_{12}I_{34}$ the pencil of planes on p_5p_6 is invariant in such wise that the plane on p_i $(i=1,\cdots,4)$ is transformed into itself. Hence every plane E of the pencil is invariant, and in E I_{1234} is a quadratic transformation T with direct and inverse F-points at p_5, p_6 and with three corresponding pairs at the points where E cuts opposite edges of the tetrahedron, $p_1,\cdots,p_4$. This tetrahedron cuts E in a four-line and the pencil of conic envelopes on the four lines determines at any point of E an involution of pairs of tangents which contains the pairs of lines to the opposite points mentioned. The two involutions at points p_5 and p_6 of E construct the quadratic transformation T on E. One conic of the pencil touches the line p_5p_6 and the third F-point is the meet of the two further tangents to this conic form p_5, p_6. The four fixed points of T are the meets of the two tangents at p_5 to the two conics of the pencil on p_5 with the two tangents at p_6 to the two conics on p_6. Hence the locus of fixed points of I_{1234} is cut by a variable plane E on p_5p_6 in a variable set of four points.

Under I_{1234} the net of lines on p_6 is invariant. Projecting from p_6, the points $p_1,\cdots,p_5$ pass into a P_5^2 in S_2 and I_{1234} in S_2 determines the transformation I_{1234} on the net of lines on p_6. Hence under I_{1234} each line of a cubic cone with vertex at p_6 and on $p_1,\cdots,p_5$ is invariant. Similarly each line of a cubic cone with vertex at p_5 and on $p_1,\cdots,p_4,p_6$ is invariant. If x is on both cones it is transformed by I_{1234} both along the line p_6x, and along the line p_5x, and therefore is fixed. The curve of intersection of the

two cones, apart from the line p_5p_6, is an octavic curve with nodes at p_5, p_6 and on $p_1, \cdots, p_4$. This curve will project from either of its nodes into a doubly covered cubic curve.

Finally, as is well known, the involution $I_{123456} = I$ interchanges the planes π_{123} and π_{456}, and therefore leaves each member of the web of quadrics on $P_6{}^3$ invariant. Thus I is the involution which interchanges the seventh and eighth base points of a net of quadrics on six given points, and its locus of fixed points is the locus of nodes of quadrics of the web, i. e., the Weddle surface W_2.

The foregoing particular cases all illustrate the following theorem:

(32) *The element $I_{i_1 \ldots i_{2l}}$ of $G_2{}^{n+2}$ in S_n has a locus of fixed points of dimension $l-1$, which, if $l > 2$, is projected from the linear space F_{n+2-2l} determined by $p_{i_{2l+1}}, \cdots, p_{i_{n+3}}$ into a Weddle manifold W_{l-1} in an S_{2l-3} which is covered 2^{n+3-2l} times.*

For, in the first place, if $l = 1$, the regular transformation $I_{i_1 i_2}$ of order n has 2^n fixed points which, for the canonical form (6), have coördinates $\pm 1, \pm 1, \cdots, \pm 1$. When n is odd these 2^n fixed points divide into two symmetrical sets of 2^{n-1} each according as the number of $+$ signs in the coördinates is even or odd.

In the general case let $I_{i_1 \ldots i_{2l}}$ be represented as the product $I_{i_1 i_2} I_{i_3 i_4} \cdots I_{i_{2l-1}\, i_{2l}}$. Each of these factors has F-points at $p_{i_{2l+1}}, \cdots, p_{i_{n+3}}$ in F_{n+2-2l}. Each factor, and therefore also $I_{i_1 \ldots i_{2l}}$, transforms the linear system of spaces E_{n+3-2l} on F_{n+2-2l} into itself [cf. (9)]. If the points $p_{i_1}, \cdots, p_{i_{2l}}$ are projected from F_{n+2-2l} into a set $q_{i_1}, \cdots, q_{i_{2l}}$ in an S_{2l-3}, the involution $I_{i_1 \ldots\, {}_{2l}}$ in S_{2l-3} has a locus of fixed points, W_{l-1}. Hence, under $I_{i_1 \ldots i_{2l}}$ in S_n, ∞^{l-1} of the spaces E are fixed. If x is a fixed point of $I_{i_1 \ldots i_{2l}}$, the space E on x must be fixed. Let E be a particular fixed space on F_{n+2-2l}, on which $I_{i_1 \ldots i_{2l}}$ effects the transformation T. This E is transformed by $I_{i_1 i_2}$ into $E_{i_1 i_2}$, $E_{i_1 i_2}$ is transformed by $I_{i_2 i_4}$ into $E_{i_3 i_4}$, etc.; finally $E_{i_{2l-3}\, i_{2l-2}}$ is transformed by $I_{i_{2l-1} i_{2l}}$ back into E. According to (9) these constituent transformations of E' into E'' are regular and of order $n+3-2l$. Each has the fixed set, $p_{i_{2l+1}}, \cdots, p_{i_{n+3}}$, of $n+3-2l$ F-points. Hence the product T is a transformation of the same order $n+3-2l$ in E, which has a set of 2^{n+3-2l} fixed points, all of which project into the same point of W_{l-1}.

§ 2

A Transcendental Definition of W_p Suggested by the Planar Hyperelliptic Curve.

7. The manifold W_p and the planar hyperelliptic curve H_p. The theory of the hyperelliptic plane curve, H_p^{p+2}, of order $p+2$ and genus p with a p-fold point at O, under Jonquières transformation of the pencil of lines on O into itself is found in ([1] pp. 125-133). The curve possesses a g_1^2, cut out by the lines on O, whose branch points, $r_1, \cdots, r_{2p+2}$ are the contacts of tangents from O. We follow the exposition cited except that the branch point r_1, which figures [cf. [1] p. 131 (1)] as the fixed lower limit of the p normal integrals of the first kind, is here replaced by r_{2p+2}.

The planar set of points, R^2_{2p+3}, consisting of $r_1, \cdots, r_{2p+2}$ and $O = r_{2p+3}$, satisfies $p-1$ projective conditions [cf. [1] p. 127 (9)], and therefore depends upon $(4p+6)-8-(p-1)=3p-1$ absolute projective constants. On the other hand, it is well known that two curves H_p, whose $2p+2$ branch lines on O are projectively equivalent, are themselves birationally equivalent; and thus H_p depends on only $2p-1$ absolute birational constants. Two curves, H_p, H_p' which are birationally equivalent, but not projectively equivalent, can be transformed into each other by a Cremona transformation of that particular type known as a Jonquières transformation J [cf. [1] p. 128 (12)]. If on H_p a p-ad of points be chosen, J can always be determined so that this p-ad is converted into the p-fold point O' of H_p'. As this p-ad varies on H_p the ∞^p curves H_p', all projectively distinct but birationally equivalent, are obtained. We denote the parameters of the $2p+2$ branch lines of H_p on O by $t_1, t_2, \cdots, t_{2p+2}$. These may be regarded as fixed for the ∞^p curves H_p'.

The planar set of points, R^2_{2p+3}, is "associated" with a set of $2p+3$ points P^{2p-1}_{2p+3} in a space S_{2p-1} in such wise that the points $p_1, \cdots, p_{2p+2}$, and $x = p_{2p+3}$, of P^{2p-1}_{2p+3} correspond respectively to the points $r_1, \cdots, r_{2p+2}$, and $O = r_{2p+3}$, of R^2_{2p+3} (cf. [1] pp. 42-45). In S_{2p-1} the parameters of $p_1, \cdots, p_{2p+2}$ on the rational norm-curve, N^{2p-1}, determined by them, are projective to the lines of the pencil in S_2 from O to $r_1, \cdots, r_{2p+2}$ (cf. [1] p. 43 (c)). These parameters may therefore be taken to be $t_1, \cdots, t_{2p+2}$. Thus as H_p varies through the system of ∞^p curves H_p', the set P^{2p-1}_{2p+3} associated with R'^2_{2p+3} has a set of $2p+2$ points fixed at $p_1, \cdots, p_{2p+2}$ and a variable point x which takes up ∞^p positions. We wish to prove that the locus of x is the manifold W_p, the locus of fixed points of the element $I = I_{1,2,\ldots,2p+2}$ of **5**.

The conditions on the planar set R^2_{2p+3} that its points be the branch points

and multiple point of an H_p are the conditions that a Jonquières transformation J^{p+2} of order $p+2$ exist for which the set is congruent to itself in the identical order. In fact, if H_p exists, the J^{p+2} is precisely that Cremona involution which has H_p for a locus of fixed points. Moreover this J^{p+2} can be expressed as the product (cf. [1] p. 129),

$$(33) \qquad J^{p+2} = A_{O12}(12) \cdot A_{O34}(34) \cdot \ldots \cdot A_{O,2p+1,2p+2}(2p+1, 2p+2);$$

where it is understood that A_{O12} represents the quadratic involution with F-points at O, r_1, r_2, and with respective corresponding P-loci, r_1r_2, Or_2, Or_1; and that (12) represents the permutation of r_1, r_2. Thus $A_{O12}(12)$ represents the perspective quadratic involution for which every line on O is fixed.

If $R^2{}_{2p+3}$ is congruent to a set of points $R'^2{}_{2p+3}$ under A_{O12} (cf. [1] p. 10), its associated set P^{2p-1}_{2p+3} is congruent to a set of points P'^{2p-1}_{2p+3} under the regular transformation $A_{3,4,\ldots,2p+2}$ of order $2p-1$ in S_{2p-1} (cf. [1] p. 40 and p. 45 (8)) with F-points at $p_3, \cdots, p_{2p+2}$. Also the transposition (12) of r_1, r_2 entails the transposition of p_1, p_2 in S_{2p-1}. Hence to the transformation $A_{O12}(12)$ in S_2 there corresponds in S_{2p-1} the element I_{12} of the group $G_2{}^{n+2}$. If $R^2{}_{2p+3}$ is self-congruent in the identical order under the product (33), then P^{2p-1}_{2p+3} is self-congruent in the identical order under $I = I_{12}I_{34} \cdots I_{2p+1,2p+2}$; i. e., x of the set P^{2p-1}_{2p+3} is a fixed point of I. Conversely, if x is a fixed point of $I = I_{12} \cdots I_{2p+1,2p+2}$, the set P^{2p-1}_{2p+3} is self-congruent under I, and in S_2 the associated set $R^2{}_{2p+3}$ is self-congruent under the product (33). Hence $(J^{p+2})^2$ is a collineation which may be taken to be the identity, since only projective distinctions are being observed. Then J^{p+2} is involutorial, and has a curve H_p of fixed points for which $R^2{}_{2p+3}$ is the set of branch points and the multiple point. Hence

(34) *The locus of fixed points of the element I of $G_2{}^{n+2}$ $(n = 2p-1)$ is a manifold W_p of dimension p determined by the set P^{2p-1}_{2p+2} of points whose parameters on their norm-curve N^{2p-1} are $t_1, \cdots, t_{2p+2}$. As x varies on W_p the set $P^{2p-1}_{2p+3} = P^{2p-1}_{2p+2}$, x is associated with the sets $R'^2{}_{2p+3}$ of branch points and multiple point of the ∞^p projectively distinct hyperelliptic curves H_p' which have a fundamental $(2p+2)$-ic with roots projective to $t_1, \cdots, t_{2p+2}$.*

The $p-1$ projective conditions on the planar set $R^2{}_{2p+3}$ determined by H_p lead to geometric properties of the point x on W_p which are just sufficient to confine it to W_p. For example, if $p=2$, $r_1, \cdots, r_6$ are on a conic. But $r_1, \cdots, r_6$ are associated with the projection of $p_1, \cdots, p_6$ from x on the Weddle surface. Hence these projected points are likewise on a conic; i. e., x is the vertex of a quadric cone on $p_1, \cdots, p_6$ (cf. [1] p. 134). If

$p=3$, $r_1, \cdots, r_8$ and O are the base points of a pencil of cubics ([1] p. 127 (9)), and if three of the nine points are on a line, the complementary six are on a conic. Hence conics map the plane on a Veronese surface, V_2^4, and map the nine points, R_9^2, upon the associated P_9^5 in S_5. They also map the pencil of cubics on R_9^2 into a pencil of normal elliptic sextics, E^6, on P_9^5, which pencil describes V_2^4. Thus a point x of W_8 forms with P_8^5 the set of nine base points of ∞^1 E^6's; whereas a generic point of S_5 together with P_8^5 is on a unique E^6 (cf. [7] pp. 16-18). The V_2^4 is the locus of nodes of a cubic spread V_4^3; and thus there is isolated at x a particular member of the linear system Σ of (31a). The corresponding particular member, in the case $p=2$, is the nodal quadric on P_6^3 with node at x on W_2.

8. The loci on W_p which correspond to discriminant conditions on the planar set R^2_{2p+3}. If R^2_{2p+3} is subjected to a projective condition, not dependent on the $p-1$ conditions which it already satisfies, the associated set P^{2p-1}_{2p+3} in S_{2p-1} is also subjected to an additional projective condition. This additional condition may fall, as we shall see, upon the points of P^{2p-1}_{2p+2} alone; but in general it will also involve the point x of W_p, and will, for given P^{2p-1}_{2p+2}, confine x to a locus on W_p of dimension $p-1$.

It is clear that those conditions on R^2_{2p+3}, which are most intimately related to the theory of the set under Cremona transformation, will be of greatest importance. Such are the "discriminant conditions" of the set R^2_{2p+3} (cf. [1] p. 42). They are the conjugates under Cremona transformation of the condition that some two points of the set coincide. Nevertheless, in forming these conjugates, we use, not the general Cremona transformation with F-points in the set R^2_{2p+3}, but rather the Jonquières types with center at O and simple points in the set $r_1, \cdots, r_{2p+2}$. For it is only these latter types which convert an H_p into an H_p'. Thus we apply to these conditions only a sequence of quadratic transformations $A_{Or_ir_j}$.

Under these circumstances we find (cf. [1] p. 130) two conjugate sets of discriminant conditions. The first set arises from the coincidence of r_i and r_j, and is composed of $(p+1)(2p+1)$ pairs of conditions, $\delta_{ij}=0$ and $\delta(Or_ir_j)^1=0$, which indicate respectively that r_i and r_j coincide, and that O, r_i, r_j are on a line. Then either H_p acquires a node at $r_i=r_j$, or the line $(Or_ir_j)^1$ factors out from H_p; and the genus p is reduced.

The second conjugate set arises from a coincidence of r_i and O, which occurs when one of the p branches at O has a flex-point at O. The conjugates of this are of the form $\delta(O^k r_1 \cdots r_{2k+3})^{k+1}=0$ $(k=-1, 0, \cdots, p-1)$ which represents the condition that there exists a curve of order $k+1$ with k-fold point at O and on $2k+3$, rather than only $2k+2$, of the branch

points; except that, for $k = -1$, it represents the coincidence of O and r_1. Then (cf. [1] p. 130 (17)).

(35) *The set R^2_{2p+3} under Jonquières transformation has a conjugate set of 2^{2p+1} discriminant conditions which divide into 2^p pairs,*

$$\delta(O^k r_1 \cdots r_{2k+3})^{k+1} = 0, \quad \delta(O^{p-k-2} r_{2k+4} \cdots r_{2p+2})^{p-k-1} = 0$$
$$(k = -1, 0, \cdots, p-1),$$

either of which implies the other.

In fact, if one condition of a pair is satisfied, the existence of the Jonquières involution J^{p+2} defined by H_p implies that the other condition of the pair must also be satisfied.

We now translate these discriminant conditions on R^2_{2p+3} to the corresponding discriminant conditions on the associated set P^{2p-1}_{2p+2} in S_{2p-1}. This is accomplished by observing that if two points of the one set coalesce, the corresponding two points of the associated set also coalesce; and that to the quadratic perspective element $A_{Or_1r_2}(12)$ in the plane there corresponds in S_{2p-1} the element I_{12} of the group $G_{2^{p+2}}$ (cf. [1] p. 45 (8), (9)). If then, in R^2_{2p+3}, r_1 and O coincide while, in P^{2p-1}_{2p+3}, p_1 and x coincide, we find, by applying the corresponding elements $A_{Or_2r_3}(23)$ and I_{23}, that $\delta(r_1 r_2 r_3)^1 = 0$ implies that x on W_p is on the P-locus π_{123}. Indeed the coincidence of x with p_1 means that x is on the P-locus π_1 and π_1 is transformed by I_{23} into π_{123} (cf. (14)). Continuing in this way we find that

(36) *The 2^p pairs of discriminant conditions* (35) *imply for the associated set P^{2p-1}_{2p+3} that x of W_p is on the section of W_p by either one of the pair of P-loci*

$$\pi_{1,2,\ldots,2k+3}, \quad \pi_{2k+4,\ldots,2p+2}.$$

That each P-locus of a pair cuts W_p in the same manifold is immediately evident from the fact that W_p is a locus of fixed points under I, and that the two P-loci are interchanged by I (cf. (31a)).

Reverting to the first type of discriminant condition on R^2_{2p+3}, we observe that $\delta_{ij} = 0$ implies in S_{2p-1} the coincidence of p_i, p_j; i. e., a double root of the fundamental $(2p+2)$-ic which determines the set P^{2p-1}_{2p+2} on the norm-curve N^{2p-1}. Also $\delta(Or_ir_j)^1 = 0$ implies that all of the $2p+2$ points p except p_i, p_j are on an S_{2p-2}. Hence N^{2p-1} must degenerate into an N^{2p-2} containing $2p$ of the points, and into the line p_ip_j which must meet N^{2p-2} in a point p. On N^{2p-2} the fundamental $(2p+2)$-ic consists of the $2p$ points and of p counted twice. These special cases of lower genus are not

considered further as we assume throughout that the fundamental $(2p+2)$-ic has distinct roots.

9. The equations in terms of theta functions of the sections of W_p by the P-loci of P^{2p-1}_{2p+2}. We recall some parts of the transcendental theory of H_p as given in ([1] pp. 131-133) with the single change that the common lower limit r_1 of the normal integrals of the first kind attached to H_p is here replaced by r_{2p+2}.

If $u = u_1, \cdots, u_p$ denote the sums of the respective normal integrals of the first kind with upper limits at a p-ad of points on H_p, then in general the p-ad determines $u_1, \cdots, u_p$ and conversely. The superposed p-ad, the correspondent of the given p-ad in the $g_1{}^2$ on H_p, then determines $-u$. The given p-ad and the superposed p-ad determine the same projective form of H_p, being interchanged by any Cremona involution (cf. [1] p. 126 (7)) which effects on H_p the involution whose pairs are in $g_1{}^2$.

The ∞^p projectively distinct curves H_p' which are birationally equivalent to H_p are distinguished by the choice of the p-ad of points on H_p which passes into the multiple point on H_p'. Thus the ∞^p points of W_p (cf. (34)) can be named by the sets of values $\pm u$, each determined by a p-ad of points on H_p (or by its superposed p-ad).

The P-curves of the planar set $R^2{}_{2p+3}$ which correspond under Jonquières transformation to directions at O are 2^{2p+1} in number, and divide into 2^{2p} pairs (cf. [1] p. 129 (15), (16)),

$$(37) \qquad L_k(r_1, \cdots, r_{2k+2}), \quad L_{p-k-1}(r_{2k+3}, \cdots, r_{2p+2}) \qquad (k = -1, 0, \cdots, p),$$

where L_k is a curve of order $k+1$ with a k-fold point at O and on $r_1, \cdots, r_{2k+2}$, and L_{-1} is the set of directions at O. The two curves of a pair cut out on H_p the same additional p-ad, whose parameters u' are related to u, the parameters of the p-ad at O, by the equations (cf. [1] p. 132 (7)):

$$(38) \qquad u' \equiv u + P_{1,2,\ldots,2k+2} \equiv u + P_{2k+3,\ldots,2p+2},$$

where $P_{1,2,\ldots,2j}$ is a half-period of the hyperelliptic theta functions in the basis notation. Since the p-ad at O is converted into the p-ad on L_k by the Jonquières transformation $A_{O12}(12) \cdots A_{O,2k+1,2k+2}(2k+1, 2k+2)$, the point x on W_p, which corresponds to the p-ad at O, is converted into the point x', which corresponds to the p-ad on L_k, by the element $I_{12} \cdots I_{2k+1,2k+2} = I_{1,2,\ldots,2k+2}$ of $G_2{}^{n+2}$. This element effects on W_p the same transformation as $I_{2k+3,\ldots,2p+2}$. Hence

(39) *If the points x of W_p are determined transcendentally by the value systems $\pm u = \pm u_1, \cdots, \pm u_p$, the factor group g_2^{2p} of G_2^{p+2} under I* (cf. (31a)) *effects on W_p the transformations expressed by the group* (38) *of additive half-periods.*

With the proper half periods represented as in (38) by the sets of $2k+2$, or the complementary set of $2(p-k)$ $(k=0,1,\cdots,p-1)$, indices selected from $2p+2$ indices, the 2^{2p} odd and even theta functions of the first order are represented in this basis notation by the sets of $p+1-2k$, or the complementary sets of $p+1+2k$, indices; i. e., they are

$$\vartheta_{1,2,\ldots,p+1-2k}(u) = \vartheta_{p+2-2k,\ldots,2p+2}(u) \quad [k=0,1,\cdots,(p+1)/2]. \tag{40}$$

They are even or odd according as k is even or odd. If then p is odd there is a function $\vartheta(u)=\vartheta_{1,2,\ldots,2p+2}(u)$ which is even or odd according as $p \equiv 3$ or 1 (mod. 4). The hyperelliptic thetas differ from the general thetas in that all of the even functions (40), for which $k \neq 0$, vanish for the zero argument (cf. [8] pp. 456-464).

In ([1] p. 133) there is given a proof that the satisfaction of one of the 2^{2p} distinct discriminant conditions (35) entails the vanishing of a theta function for the arguments u of the p-ad at O on H_p. According to (36) these are also the conditions that x on W_p lie on a P-locus of the set P^{2p-1}_{2p+2}. We can therefore state at once that

(41) *If p is even, the condition that x on W_p with parameters u is on the P-locus $\pi_{1,2,\ldots,2k+3}$, or on $\pi_{2k+4,\ldots,2p+2}$ is*

$$\vartheta_{1,2,\ldots,2k+3}(u)=\vartheta_{2k+4,\ldots,2p+2}(u)=0.$$

If p is odd, the condition that x is on the same pair of P-loci is

$$\vartheta_{2k+4,\ldots,2p+1}(u)=\vartheta_{1,2,\ldots,2k+3,2p+2}(u)=0.$$

10. A provisional parametric representation of W_p. In giving a parametric representation of W_p by means of the hyperelliptic theta functions, it would suffice to give the section of W_p by $2p$ linear spaces which form a $2p$-edron in S_{2p-1}. But greater symmetry is obtained by giving the sections cut out by all the linear spaces on $2p-1$ points of the set P^{2p-1}_{2p+2}. These are the sections of W_p by the P-loci $\pi_{i_1 i_2 i_3}$ $(i_1, i_2, i_3 = 1, \cdots, 2p+2)$. According to (41) the parameters u of a generic point on this section constitute a zero of a definite theta function. But this particular linear section is characterized also by the fact that it passes through all the points of P^{2p-1}_{2p+2}

except the points p_{i_1}, p_{i_2}, p_{i_3}. This behavior of the section is naturally indicated by the fact that its parametric representation contains factors which vanish when those discriminant conditions are satisfied, which imply a coincidence of x with a point p other than the three mentioned. The cases, p even, p odd, have somewhat different behavior.

When p is even, the directions at the points of P^{2p-1}_{2p+2} are those of π_{i_1}, and, according to (41), these are determined by $\vartheta_{i_1}(u)=0$. If then we set

$$(42) \qquad p \text{ even}: \quad \pi_{i_1 i_2 i_3} = \vartheta_{i_1 i_2 i_3} \cdot \Pi / \vartheta_{i_1}\vartheta_{i_2}\vartheta_{i_3},$$
$$\Pi = \vartheta_{i_1}\vartheta_{i_2}\cdots\vartheta_{i_{2p+2}}$$

we have a parametric representation of a p-way (u) in terms of theta products of order $2p$ and characteristic zero, which has the same behavior as W_p.

When p is odd, the index $2p+2$ is isolated. The directions at $p_1, \cdots, p_{2p+1}$ are determined respectively by $\vartheta_{1,2p+2}(u)=0, \cdots, \vartheta_{2p+1,2p+2}(u)=0$, while the directions at p_{2p+2} are determined by $\vartheta(u)=0$. The generic point on the section $\pi_{i_1, i_2 2p+2}$ is determined by $\vartheta_{i_1 i_2}(u)=0$. The generic point on the section $\pi_{i_1 i_2 i_3}$ $(i_j \neq 2p+2)$ is determined by $\vartheta_{i_1, i_2, i_3, 2p+2}(u)=0$. The parametric representation is then

$$p \text{ odd}: \quad \pi_{i_1, i_2, 2p+2} = \vartheta_{i_1 i_2} \cdot \Pi / \vartheta\vartheta_{i_1, 2p+2}\vartheta_{i_1, 2p+2},$$
$$(43) \qquad \pi_{i_1 i_2 i_3} = \vartheta_{i_1 i_2 i_3, 2p+2} \cdot \Pi / \vartheta_{i_1, 2p+2}\vartheta_{i_2, 2p+2}\vartheta_{i_3, 2p+2},$$
$$\Pi = \vartheta\vartheta_{1, 2p+2}\vartheta_{2, 2p+2}\cdots\vartheta_{2p+1, 2p+2}.$$

Again the p-way is given in terms of functions of order $2p$ and characteristic zero.

In order to prove that the representations (42) and (43) are precisely those of the p-way, W_p, it is necessary to prove first that the functions to which the linear P-loci are equated satisfy the same linear relations as the P-loci. Secondly the coefficients of the linear relations satisfied by these functions must be such that the linear P-loci define a set of points projective to P^{2p-1}_{2p+2}. Thirdly the new functions which arise when u is replaced by $u+P_{12}$ (P_{12} a half period) must have a rational representation in terms of the original functions which is precisely that of the involution I_{12} of the Cremona $G_{2^{p+2}}$. The p-way will then necessarily be a locus of fixed points of the symmetrical element $I = {}_{1,2,\ldots,2p+}$ of $G_{2^{p+2}}$, and will coincide with W_p. These three requirements are disposed of in the following section.

§ 3

TRANSCENDENTAL DEFINITION OF W_p.

11. Three-term hyperelliptic theta relations. We divide the $2p+2$ indices of the basis notation into complementary sets of four, a, b, c, d; and of $2p-2$, $i_1 \cdots i_{2p-2}$. If to the argument of any one of the 2^{2p} odd and even theta functions of the first order we add the half period P_{abcd}, this function is converted, to within an outstanding exponential factor, into another one of the set, such that the product of the two is a theta function of second order and characteristic $abcd$. Of these 2^{2p-1} products half are even and half are odd. Of the $2^{2(p-1)}$ odd (or even) products only 2^{p-1} are linearly independent ([1] p. 56 (9)).

The odd products have the following typical forms:

$$p = 2k: \quad \vartheta_{i_1 i_2 \ldots i_{2s-2} a}\vartheta_{i_1 i_2 \ldots i_{2s-2} bcd} \qquad (s = 1, \cdots, k);$$
$$p = 2k+1: \quad \vartheta_{i_1 i_2 \ldots i_{2s-1} a}\vartheta_{i_1 i_2 \ldots i_{2s-1} bcd}.$$

The theorem which we wish to prove reads as follows:

(44) *Any three of the following four products are linearly related:*

(a) p even: $\vartheta_{i_1}\vartheta_{i_2 i_3 i_4}$, $\vartheta_{i_2}\vartheta_{i_1 i_3 i_4}$, $\vartheta_{i_3}\vartheta_{i_1 i_2 i_4}$, $\vartheta_{i_4}\vartheta_{i_1 i_2 i_3}$;

(b) p odd: $\vartheta_{i_1 i_5}\vartheta_{i_2 i_3 i_4 i_5}$, $\vartheta_{i_2 i_5}\vartheta_{i_1 i_3 i_4 i_5}$, $\vartheta_{i_3 i_5}\vartheta_{i_1 i_2 i_4 i_5}$, $\vartheta_{i_4 i_5}\vartheta_{i_1 i_2 i_3 i_5}$;

(c) p odd: $\vartheta_{i_1 i_4}\vartheta_{i_2 i_3}$, $\vartheta_{i_2 i_4}\vartheta_{i_1 i_3}$, $\vartheta_{i_3 i_4}\vartheta_{i_1 i_2}$, $\vartheta\vartheta_{i_1 i_2 i_3 i_4}$

$$(i_1, \cdots, i_5 = 1, \cdots, 2p+2).$$

For, any $2^{p-1}+1$ of the $2^{2(p-1)}$ odd (or even) products must satisfy at least one linear relation, $R = 0$, with constant coefficients which are not all zero. If the $2^{p-1}+1$ products can be arranged in a certain sequence, and if to each product, beginning with the first and continuing up to a certain point, there can be assigned a half period such that the product does not vanish for its assigned half period whereas the succeeding functions vanish for the half period, then there must exist a relation among the products to which such half periods have not been assigned. For, the substitution of the first half period in $R = 0$ yields a zero in every term except the first, whence the coefficient of the first function is zero. Similarly by substituting the second half period in the simplified relation we find that the second function cannot appear, etc. We recall that the half period $P_{a_1 \ldots a_{2l}}$ substituted in the function $\vartheta_{\beta_1 \ldots \beta_j}(u)$ yields $E \cdot \vartheta_{a_1 \ldots a_{2l} \beta_1 \ldots \beta_j}(0)$, where E is a non-vanishing

exponential; and that $\vartheta_{\alpha_1 \dots \alpha_{2l}\beta_1 \dots \beta_j}(0) = 0$ except when the indices $\alpha_1 \cdots \alpha_{2l}\beta_1 \cdots \beta_j$, with repeated indices deleted, number $p+1$.

When $p = 2k$ we divide the $2p+2$ indices into three complementary sets $i_1, \cdots, i_{2k-1}$; $j_1, \cdots, j_{2k-1}$; a, b, c, d; and form the following sequence of products:

$$\begin{array}{ll} \vartheta_{i_1 \dots i_{2k-2}a}\vartheta_{i_1 \dots i_{2k-2}cbd}; & \vartheta_{i_1 \dots i_{2k-2}b}\vartheta_{i_1 \dots i_{2k-2}acd}; \\ \vartheta_{i_1 \dots i_{2k-4}a}\vartheta_{i_1 \dots i_{2k-4}bcd}; & \vartheta_{i_1 \dots i_{2k-4}b}\vartheta_{i_1 \dots 2k-4acd}; \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \end{array}$$

$$\vartheta_{i_1 i_2 a}\vartheta_{i_1 i_2 bcd}; \qquad \vartheta_{i_1 i_2 b}\vartheta_{i_1 i_2 acd};$$

$$\vartheta_a\vartheta_{bcd},\ \vartheta_b\vartheta_{acd},\ \vartheta_c\vartheta_{abd}.$$

It is to be understood that $\vartheta_{i_1 \dots i_{2k-2}a}\vartheta_{i_1 \dots i_{2k-2}bcd}$ represents the aggregate of products for the $\binom{2k-1}{2k-2}$ choices of $i_1, \cdots, i_{2k-2}$ from the set $i_1, \cdots, i_{2k-1}$; etc. The order of precedence in such an aggregate is not material. An aggregate on the right follows one on the left. The number of products is

$$2\left\{\binom{2k-1}{2k-2}+\binom{2k-1}{2k-4}+\cdots+\binom{2k-1}{2}\right\}+3$$
$$=(1+1)^{2k-1}-(1-1)^{2k-1}+1=2^{p-1}+1.$$

To this sequence of products we adjoin a sequence of half periods:

$$\begin{array}{ll} P_{i_{2k-1}b}; & P_{i_{2k-1}a}; \\ P_{i_{2k-3}i_{2k-2}i_{2k-1}b}; & P_{i_{2k-3}i_{2k-2}i_{2k-1}a}; \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ P_{i_3 \dots i_{2k-1}b}; & P_{i_3 \dots i_{2k-1}a}, \end{array}$$

one for each product except the last three. An inspection of the two sequences shows that each half period is not a zero of its corresponding product and is a zero of each later product. Hence the last three products are linearly related which, for $a, b, c, d = i_1, \cdots, i_4$, proves (44a).

When $p = 2k+1$ we divide the $2p+2$ indices into three complementary sets, $i_1, \cdots, i_{2k}$; $j_1, \cdots, j_{2k}$; and a, b, c, d. The following sequence of products and attached half periods serves the purpose:

$$\begin{array}{lcl} \vartheta_{i_1 \dots i_{2k-1}a}\vartheta_{i_1 \dots i_{2k-1}bcd} & : & P_{i_{2k}b}; \\ \vartheta_{i_1 \dots i_{2k-2}j_1 b}\vartheta_{i_1 \dots i_{2k-2}j_1 acd} & : & P_{i_{2k-1}i_{2k}j_1 a}; \\ \vartheta_{i_1 \dots i_{2k-3}a}\vartheta_{i_1 \dots i_{2k-3}bcd} & : & P_{i_{2k-2}i_{2k-1}i_{2k}b}; \\ \vartheta_{i_1 \dots i_{2k-4}j_1 b}\vartheta_{i_1 \dots i_{2k-4}j_1 acd} & : & P_{i_{2k-3} \dots i_{2k}j_1 a}; \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \end{array}$$

$$\begin{aligned}
&\vartheta_{i_1 i_2 j_1 b}\vartheta_{i_1 i_2 j_1 acd} &&: \quad P_{i_2 \ldots i_{2k} j_1 a};\\
&\vartheta_{j_2 \ldots j_{2k} b}\vartheta_{j_2 \ldots j_{2k} acd} &&: \quad P_{j_1 a};\\
&\vartheta_{j_2 a}\vartheta_{j_2 bcd} &&: \quad P_{j_1 j_3 \ldots j_{2k} b};\\
&\vartheta_{j_3 a}\vartheta_{j_3 bcd} &&: \quad P_{j_1 j_2 j_4 \ldots j_{2k} b};\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
&\vartheta_{j_{2k} a}\vartheta_{j_{2k} bcd} &&: \quad P_{j_1 \ldots j_{2k-1} b};
\end{aligned}$$

$$\vartheta_{j_1 a}\vartheta_{j_1 bcd},\ \vartheta_{j_1 b}\vartheta_{j_1 acd},\ \vartheta_{j_1 c}\vartheta_{j_1 abd}.$$

Again the first products represent an aggregate determined by the choice of the indices i, but the indices j, a, b, c, d are fixed as given. Thus the number of products is

$$\binom{2k}{2k-1}+\binom{2k}{2k-2}+\cdots+\binom{2k}{2}+\binom{2k}{1}+3=(1+1)^{2k}+1=2^{p-1}.$$

The behavior of the products with respect to the attached half periods is the same as before whence there exists a linear relation among the last three which proves (44b).

If, in the relations (44b), u is replaced by $u+P_{i_4 i_5}$, the relations (44c) appear.

12. Transcendental definition of $Q^n{}_{n+3}$, w_p, and w_p'. We return to a consideration of the $\binom{2p+2}{3}$ theta products of order $2p$, $\pi_{i_1 i_2 i_3}$, which are defined in (42), (43). There follows from (44) that

(45) (a) *Any three of the four functions,* $\pi_{i_2 i_3 i_4}$, $\pi_{i_1 i_3 i_4}$, $\pi_{i_1 i_2 i_4}$, $\pi_{i_1 i_2 i_3}$, *are linearly related.* (b) *Of the* $\binom{k}{3}$ *functions* $\pi_{i_1 i_2 i_3}$ $(i_1, i_2, i_3 = 1, \cdots, k)$ *only* $k-2$ *are linearly independent. A typical independent set is* $\pi_{i_1 i_2 i_3}$, $\pi_{i_1 i_2 i_4}$, $\cdots$, $\pi_{i_1 i_2 i_k}$.

For, if p is even, (45a) is an immediate consequence of (44a). If p is odd, it is a consequence of (44b) or (44c) according as the index $2p+2$ is not, or is, included among $i_1, \cdots, i_4$. Then (45b) is true for $k=4$, and we assume that it is true for values up to and including $k-1$. Given then $\pi_{i_1 i_2 i_k}$ in addition to $\pi_{i_1 i_2 i_j}$ $(j=3, \cdots, k-1)$, $\pi_{i_1 i_j i_k}$, and similarly $\pi_{i_1 i_h i_k}$ $(h \neq j)$, can be obtained linearly. Hence $\pi_{i_j i_h i_k}$ can be obtained when $\pi_{i_1 i_2 i_k}$ is added to the set $\pi_{i_1 i_2 i_3}, \cdots, \pi_{i_1 i_2 i_{k-1}}$, and (45b) is true for the value k. From this there follows that

(46) *Of the* $\binom{2p+2}{3}$ *functions* $\pi_{i_1 i_2 i_3}$ *only* $2p$ *are linearly independent. If* $2p$ *independent functions are equated to the homogeneous coördinates of a point* x *in* S_{2p-1}, *then for variable* u *this point* x *ranges over a variety* w_p *in* S_{2p-1} *of which the functions* $\pi_{i_1 i_2 i_3}$ *define linear sections. Of the* $\binom{2p+1}{3}$ *func-*

tions $\pi_{i_1 i_2 i_3}$ which omit a fixed index i_j $(j = 1, \cdots, 2p+2)$ only $2p-1$ are independent. They determine linear sections of w_p by spaces S_{2p-2} which are on a point q_{i_j}. The function $\pi_{i_1 i_2 i_3}$ defines the linear section of w_p by the S_{2p-2} on all the points q_{i_j} except q_{i_1}, q_{i_2}, q_{i_3}. Similarly $2p-1$ independent functions of the set $\pi_{i_1 i_2 i_3}$ which omit the index $2p+2$ define a variety w_p', and $2p+1$ points q'_{i_j} $(i_j = 1, \cdots, 2p+1)$, in S_{2p-2}, which is the projection of w_p and q_{i_j} in S_{2p-1} from the point q_{2p+2}.

Equations (42), (43) are parametric equations of the variety w_p. To obtain similar equations for w_p' we merely delete those linear spaces $\pi_{i_1 i_2 i_3}$ which contain the index $2p+2$. We have thus defined in S_{2p-1} a set of $2p+2$ points q_{i_j}, and in S_{2p-2} a set of $2p+1$ points q'_{i_j}; i. e., in S_n a set $Q^n{}_{n+3}$. For odd n the set is identified in 14 with the set of § 1 whose parameters $t_1, \cdots, t_{2p+2}$, on the N^{2p-1} containing the set, are projective to the roots of the fundamental $(2p+2)$-ic which defines the functions.

13. Transcendental equations of the P-loci of the set $Q^n{}_{n+3}$ of w_p and w_p'. We seek expressions, corresponding to the $\pi_{i_1 i_2 i_3}$ of (42) and (43), for the sections of w_p in S_{2p-1} by the entire system, $\pi_{i_1 \ldots i_{2k+1}}$, of P-loci of Q^{2p-1}_{2p+2}. These are as follows:

(47) *When the coördinates x of a point of w_p are taken in such wise that the linear sections are given by* (42) *and* (43), *then the section of w_p by the P-locus $\pi_{i_1 \ldots i_{2k+1}}$ of Q^{2p-1}_{2p+2}, of order k, is given by*

$$p \text{ even}: \quad \pi_{i_1 \ldots i_{2k+1}} = \vartheta_{i_1 \ldots i_{2k+1}} \cdot \Pi^k / \vartheta_{i_1} \vartheta_{i_2} \cdots \vartheta_{i_{2k+1}};$$

$$p \text{ odd}: \quad \pi_{i_1 \ldots i_{2k+1}} = \vartheta_{2p+2, i_1 \ldots i_{2k+1}} \cdot \Pi^k / \vartheta_{i_1, 2p+2} \cdots \vartheta_{i_{2k+1}, 2p+2};$$

$$\pi_{2p+2, i_1 \ldots i_{2k}} = \vartheta_{i_1 \ldots i_{2k}} \cdot \Pi^k / \vartheta\vartheta_{i_1, 2p+2} \cdots \vartheta_{i_{2k}, 2p+2}.$$

The proof is given here only for the more symmetric case when p is even. If in (44a) u is replaced by $u + P_{i_1 i_5}$ the three functions,

$$\text{(a)} \qquad \vartheta_{i_5} \vartheta_{i_1 \ldots i_5}, \quad \vartheta_{i_1 i_2 i_5} \vartheta_{i_3 i_4 i_5}, \quad \vartheta_{i_1 i_3 i_5} \vartheta_{i_2 i_4 i_5},$$

appear as linearly related. Then, multiplying by Π^2 and dividing by $\vartheta_{i_1} \cdots \vartheta_{i_4} \vartheta^2{}_{i_5}$, it appears that

$$\text{(b)} \qquad \vartheta_{i_1 \ldots i_5} \cdot \Pi^2 / \vartheta_{i_1} \cdots \vartheta_{i_5}, \quad \pi_{i_1 i_2 i_5} \pi_{i_3 i_4 i_5}, \quad \pi_{i_1 i_3 i_5} \pi_{i_2 i_4 i_5},$$

are linearly related. Hence the first term must represent a quadric with nodes at $q_{i_6}, \cdots, q_{i_{2p+2}}$, and simple points at $q_{i_1}, \cdots, q_{i_4}$. By virtue of the symmetry of this term in $i_1, \cdots, i_5$, and the consequent similar representations, this quadric must have a simple point at q_{i_5} also and must coincide with the P-locus $\pi_{i_1 \ldots i_5}$. Thus we find the well known linear projective relation connecting the algebraic spreads,

(c) $$\pi_{i_1 \dots i_5}, \quad \pi_{i_1 i_2 i_5}\pi_{i_3 i_4 i_5}, \quad \pi_{i_1 i_3 i_5}\pi_{i_2 i_4 i_5},$$

a relation which might have been obtained from the linear relation connecting $\pi_{i_1 i_2 i_3}$, $\pi_{i_1 i_2 i_4}$, $\pi_{i_1 i_3 i_4}$ by applying the involution $I_{i_4 i_5}$ defined by Q^{2p-1}_{2p+2}.

By adding $P_{i_6 \dots i_{2k+1}}$ to u in (a) we find that

(d) $$\vartheta_{i_5 \dots i_{2k+1}}\vartheta_{i_1 \dots i_{2k+1}}, \quad \vartheta_{i_1 i_2 i_5 \dots i_{2k+1}}\vartheta_{i_3 i_4 i_5 \dots i_{2k+1}}, \quad \vartheta_{i_1 i_3 i_5 \dots i_{2k+1}}\vartheta_{i_2 i_4 i_5 \dots i_{2k+1}}$$

are linearly related. Multiplying by Π^{2k-2} and dividing by

$$\vartheta_{i_1} \cdot \cdot \cdot \vartheta_{i_4}\vartheta^2_{i_5} \cdot \cdot \cdot \vartheta^2_{i_{2p+2}},$$

the products,

(e) $$\pi_{i_5 \dots i_{2k+1}} \cdot [\vartheta_{i_1 \dots i_{2k+1}} \cdot \Pi^k / \vartheta_{i_1} \cdot \cdot \cdot \vartheta_{i_{2k+1}}],$$
$$\pi_{i_1 i_2 i_5 \dots i_{2k+1}}\pi_{i_3 i_4 i_5 \dots i_{2k+1}}, \quad \pi_{i_1 i_3 i_5 \dots i_{2k+1}}\pi_{i_2 i_4 i_5 \dots i_{2k+1}},$$

appear as related. On comparing this latter relation with the projective relation derived from (c) by applying the involution $I_{i_6 \dots i_{2k+1}}$, which connects

(f) $$\pi_{i_5 \dots i_{2k+1}}\pi_{i_1 \dots i_{2k+1}}, \quad \pi_{i_1 i_2 i_5 \dots i_{2k+1}}\pi_{i_3 i_4 i_5 \dots i_{2k+1}}, \quad \pi_{i_1 i_3 i_5 \dots i_{2k+1}}\pi_{i_2 i_4 i_5 \dots i_{2k+1}},$$

we obtain the result (47) for p even. For, in these relations d and f, the coefficients of the last two terms must be such that, when in (d) the factor $\vartheta_{i_5 \dots i_{2k+1}}$ appears in the first term, then in (f) the factor $\pi_{i_5 \dots i_{2k+1}}$ must appear in the first term, provided (47) is assumed for smaller values of k. The residual factors, $\vartheta_{i_1 \dots i_{2k+1}}$ and $\pi_{i_1 \dots i_{2k+1}}$ are then uniquely determined and must also correspond.

The similar statement for w_p', the projection of w_p from q_{2p+2}, in S_{2p-2} with reference to the P-loci of Q'^{2p-2}_{2p+1}, the projection of Q^{2p-1}_{2p+2} from q_{2p+2}, is obtained by considering only the P-loci whose indices do not contain $2p+2$.

14. Identification of the set Q^n_{n+3} with P^n_{n+3} of § 2. We have found in **12** that $\pi_{i_1 i_2 i_3}$ is the linear section of w_p by the S_{2p-2} on all of the points of Q^{2p-1}_{2p+2} except q_{i_1}, q_{i_2}, q_{i_3}. We wish to show that this set Q^{2p-1}_{2p+2} is projective to, and therefore, if we please, identical with the set P^{2p-1}_{2p+2} of § 2. It is necessary only to prove that the parameters t' of the points of Q on their norm-curve N'^{2p-1} are projective to the roots $t_1, \cdot \cdot \cdot, t_{2p+2}$ of the fundamental $(2p+2)$-ic which defines the curve $H_p{}^{p+2}$ of § 2 birationally, and which defines the theta functions used both in § 2 and § 3.

We denote by

(48) $$(i_1 i_2 \cdot \cdot \cdot i_k)$$

the determinant of the coördinates of an ordered set of k points in an S_{k-1}.

In particular, on an S_1 with non-homogeneous coördinate t, $(i_1 i_2)$ is $t_{i_1} - t_{i_2}$. For certain products of such differences we use the notation:

$$(49)\qquad \begin{aligned} \{i_1 i_2 i_3 \cdots\} &= (i_1 i_2)(i_1 i_3)(i_2 i_3)\cdots, \\ \{i_1 i_2 \cdots ; j_1 j_2 \cdots\} &= (i_1 j_1)(i_2 j_1)\cdots(i_1 j_2)(i_2 j_2)\cdots. \end{aligned}$$

The set P^{2p-1}_{2p+2} of § 1 with parameters $t_1, \cdots, t_{2p+2}$ on N^{2p-1} is associated with the set P^1_{2p+2} of $2p+2$ points on S_1 with coördinates $t_1, \cdots, t_{2p+2}$; whence

$$(50)\qquad (i_1 i_2 \cdots i_{2p}) = \lambda \cdot \epsilon_{i_1 i_2 \ldots i_{2p} i_{2p+1} i_{2p+2}} \cdot (i_{2p+1} i_{2p+2}),$$

where ϵ is $+1$ or -1 according as its indices are in an even or odd permutation from the natural order, and λ is a factor of proportionality.

If the reference system is so chosen that N^{2p-1} takes the canonical form

$$(51)\qquad \begin{aligned} &x_i = t^{2p-1-i} \qquad (i = 0, 1, \cdots, 2p-1), \qquad \text{then,} \\ &(i_1 \cdots i_{2p}) = \{i_1 \cdots i_{2p}\} = \lambda \cdot \epsilon_{i_1 \ldots i_{2p+2}} \cdot (i_{2p+1} i_{2p+2}). \end{aligned}$$

All of the even hyperelliptic theta functions of the first order vanish when $u = 0$ except the functions $\vartheta_{i_1 \ldots i_{p+1}} = \vartheta_{i_{p+2} \ldots i_{2p+2}}$. Thomae [9] has proved that

$$(52)\qquad \begin{aligned} c_{i_1 \ldots i_{p+1}} = c_{i_{p+2} \ldots i_{2p+2}} &= \vartheta_{i_1 \ldots i_{p+1}}(0) \\ &= \mu \cdot [\{i_1 \cdots i_{p+1}\}\{i_{p+2} \cdots i_{2p+2}\}]^{1/4} \end{aligned}$$

where μ is a factor of proportionality (cf. also Zariski,[10] pp. 321 et seq.). These values enable us to determine the coefficients in the three-term relations (44) in terms of the differences of the roots $t_1, \cdots, t_{2p+2}$. In (44a),

$$\lambda_1 \vartheta_{i_1} \vartheta_{i_2 i_3 i_4} + \lambda_2 \vartheta_{i_2} \vartheta_{i_1 i_3 i_4} + \lambda_3 \vartheta_{i_3} \vartheta_{i_1 i_2 i_4} \equiv 0,$$

let u be $P_{i_3 i_4 j_1 \ldots j_{p-1}}$, $P_{i_1 i_3 i_4 j_1 \ldots j_{p-1}}$, $P_{i_1 i_2 i_4 j_1 \ldots j_{p-1}}$ in turn. Then

$$\lambda_2 c_{i_3 i_4 j_1 \ldots j_{p-1}} c_{i_1 i_2 j_1 \ldots j_{p-1}} + \lambda_3 c_{i_2 i_4 j_1 \ldots j_{p-1}} c_{i_1 i_3 j_1 \ldots j_{p-1}} = 0, \text{ etc.},$$

where in either term a minus sign may occur (cf. [8] p. 240, VIII). But, according to (52),

$$(53)\qquad \begin{aligned} c_{i_3 i_4 j_1 \ldots j_{p-1}} c_{i_1 i_2 j_1 \ldots j_{p-1}} = \eta \cdot [(i_1 i_2)(i_3 i_4)]^{1/2} \cdot [\{j_1 \cdots j_{p-1}\}\{k_1 \cdots k_{p-1}\}]^{1/2} \\ \cdot \{i_1 \cdots i_4 ; j_1 \cdots j_{p-1} k_1 \cdots k_{p-1}\}^{1/4}, \end{aligned}$$

η being an undetermined fourth root of unity. Thus the identity is

$$(54a)\qquad \begin{aligned} A_1 \vartheta_{i_1} \vartheta_{i_2 i_3 i_4} + \eta A_2 \vartheta_{i_2} \vartheta_{i_1 i_3 i_4} + \eta' A_3 \vartheta_{i_3} \vartheta_{i_1 i_2 i_4} &= 0, \\ A_r = [(i_s i_t)(i_r i_4)]^{1/2} \qquad &(r, s, t = 1, 2, 3). \end{aligned}$$

From 44(b) similar relations are derived:

$$(54b) \quad A_1\vartheta_{i_1,2p+2}\vartheta_{i_2i_3i_4,2p+2} + \eta A_2\vartheta_{i_2,2p+2}\vartheta_{i_1i_3i_4,2p+2} + \eta' A_3\vartheta_{i_3,2p+2}\vartheta_{i_1i_2i_4,2p+2} \equiv 0;$$

$$(54b') \quad B_1\vartheta_{i_1,2p+2}\vartheta_{i_2i_3} + \eta B_2\vartheta_{i_2,2p+2}\vartheta_{i_1i_3} + \eta' B_3\vartheta_{i_3,2p+2}\vartheta_{i_1i_2} \equiv 0;$$

$$(54b'') \quad B_1\vartheta_{i_3,2p+2}\vartheta_{i_1i_2} + \eta B_2\vartheta_{i_1,2p+2}\vartheta_{i_2i_3} + \eta' B_3\vartheta\vartheta_{i_1i_2i_3,2p+2} \equiv 0,$$

where B_r is obtained from A_r by interchanging i_4 and i_{2p+2}. Of these (b) is proved like (a); (b′) is obtained from (b) by adding $P_{i_4,2p+2}$ to u and then interchanging i_4 and $2p+2$; and (b″) is obtained from (b′) by adding $P_{i_1i_2}$ to u.

Let π'_{ijk} be the theta function defined earlier with respect to the set Q^{2p-1}_{2p+2}; and set

$$G'_{ijk} = \pi'_{ijk} \cdot \{ijk\}^{1/2}.$$

Then all of the theta relations (54) take the typical form:

$$(55) \quad (i_1i_4)G'_{i_2i_3i_4} + \eta(i_2i_4)G'_{i_3i_1i_4} + \eta'(i_3i_4)G'_{i_1i_2i_4} \equiv 0$$

where $G'_{i_1i_2i_3}$ is a linear space on all of the points of Q^{2p-1}_{2p+2} except q_{i_1}, q_{i_2}, q_{i_3}.

With respect to the points P^{2p-1}_{2p+2}, let $G_{i_1i_2i_3}$ be the linear space,

$$(56) \quad G_{i_1i_2i_3} = \epsilon_{i_1\ldots i_{2p+2}} \cdot (i_4 \cdots i_{2p+2}x),$$

on all of the points except p_{i_1}, p_{i_2}, p_{i_3}. The linear spaces, thus projectively defined, satisfy the projective relations,

$$(i_2i_3i_5\cdots i_{2p+2})(i_1i_5\cdots i_{2p+2}x) + (i_3i_1i_5\cdots i_{2p+2})(i_2i_5\cdots i_{2p+2}x) + (i_1i_2i_5\cdots i_{2p+2})(i_3i_5\cdots i_{2p+2}x) \equiv 0.$$

These can be modified by using (56) and (50) to read:

$$(57) \quad (i_1i_4)G_{i_2i_3i_4} + (i_2i_4)G_{i_3i_1i_4} + (i_3i_4)G_{i_1i_2i_4} \equiv 0.$$

Hence, according to (55) and (57), corresponding double ratios of four linear spaces on $Q^n{}_{n+3}$ and $P^n{}_{n+3}$ coincide to within fourth roots of unity; and the same is true of their respective associated sets, $t_1', \cdots, t'_{2p+2}$, and $t_1, \cdots, t_{2p+2}$, on S_1. But this can happen only if the fourth roots all are unity and the two linear sets are projective. For, if t_1', t_2', t_3' and t_1, t_2, t_3 are each projected into 0, 1, ∞ the coördinates of the remaining points are $t_i' = \eta_i t_i$ $(i = 4, \cdots, 2p+2)$, where the η's are fourth roots of unity. Then also $(0, 1; t_4' t_5') = \eta(0, 1; t_4, t_5)$ which, for generic t_4, t_5, requires that $\eta = \eta_4 = \eta_5 = 1$. Since the linear sets are projective, their associated sets, $Q^n{}_{n+3}$ and $P^n{}_{n+3}$, are also projective, and may be taken as superposed. Hence

(58) *The set of points $Q^n{}_{n+3}$ defined in* **12** *by the use of the functions $\pi_{i_1i_2i_3}$*

has parameters on the unique norm-curve N^n containing the set, which are projective to the roots $t_1, \cdots, t_{2p+2}$ $(n = 2p-1)$, or to the roots $t_1, \cdots, t_{2p+1}$ $(n = 2p-2)$, of the fundamental $(2p+2)$-ic which defines the functions. This set is projectively identical with the set $P^n{}_{n+3}$ of §§ 1, 2.

We observe that the possible fourth roots of unity in the formulae (54) can be reduced, by the choice of the factor and the order of i, j, k in

$$G_{ijk} = \pi_{ijk} \cdot \{ijk\}^{1/2}, \tag{59}$$

to the square roots ± 1 which occur in the projective relations (57). Unfortunately the basis notation is not sufficiently precise to distinguish between these signs (cf. [1] p. 86). But this difficulty can be removed by a transition, as in (59), to the projective form in which the signs of the expressions used depend only on the order of the points concerned.

15. Identification of w_p, w_p' with the W_p, W_p' of § 2. Let the set of points defined by the linear spaces $\pi_{i_1 i_2 i_3}$ in (42), (43) be denoted now by $P^n{}_{n+3}$. If they are expressed in terms of a coördinate x in S_n, then (42), (43) can be solved for x to obtain explicit expressions for these coördinates as theta functions of u of order $2p$. As u varies, the point x, thus determined by u, runs over w_p. We consider the relation between the two points x, x' on w_p determined by u and $u' \equiv u + P_{i_1 i_2}$. The point x' is determined by the linear expressions, $\pi'_{i_1 i_2 i_3}$, which arise when u in (42), (43) is replaced by $u + P_{i_1 i_2}$. Hence, in the case p even, which is the only one we consider in detail, typical values of the $\pi'_{i_1 i_2 i_3}$ are:

$$\begin{aligned}
&(a) \quad \rho\pi'_{i_1 i_2 i_3} = \pm\, \vartheta_{i_3} \cdot \Pi'/\vartheta_{i_1}\vartheta_{i_2}\vartheta_{i_1 i_2 i_3},\\
&(b) \quad \rho\pi'_{i_1 i_3 i_4} = \pm\, \vartheta_{i_2 i_3 i_4} \cdot \Pi'/\vartheta_{i_2}\vartheta_{i_1 i_2 i_3}\vartheta_{i_1 i_2 i_4},\\
&(c) \quad \rho\pi'_{i_2 i_3 i_4} = \pm\, \vartheta_{i_1 i_3 i_4} \cdot \Pi'/\vartheta_{i_1}\vartheta_{i_1 i_2 i_3}\vartheta_{i_1 i_2 i_4},\\
&(d) \quad \rho\pi'_{i_3 i_4 i} = \pm\, \vartheta_{i_1 \ldots i_5} \cdot \Pi'/\vartheta_{i_1 i_2 i_3}\vartheta_{i_1 i_2 i_4}\vartheta_{i_1 i_2 i},\\
&\qquad \Pi' = \vartheta_{i_1}\vartheta_{i_2}\vartheta_{i_1 i_2 i_3} \cdots \vartheta_{i_1 i_2 i_{2p+2}}.
\end{aligned}$$

From the following form of the equations (a)

$$\rho\pi'_{i_1 i_2 i_r} = \pm (1/\pi_{i_1 i_2 i_r}) \cdot \Pi'\Pi/\vartheta_{i_1}{}^2\vartheta_{i_2}{}^2 \quad (r = 3, \cdots, 2p+2), \tag{a'}$$

we see that x' is the image of x in the involutorial regular transformation of order $2p-1$ whose $2p$ F-points are at $p_{i_3}, \cdots, p_{i_{2p+2}}$. If the factors in Π' with three indices are expressed in terms of the $\pi_{i_1 i_2 i_3}$, equations (a), (b), (c) become

$$\begin{aligned}
&(a'') \quad \rho\pi'_{i_1 i_2 i_3} = \pm (\vartheta_{i_1}\vartheta_{i_2}/\Pi)^{2p-2} \cdot \pi_{i_1 i_2 i_3}\pi_{i_1 i_2 i_4} \cdots \pi_{i_1 i_2 i_{2p+2}}/\pi_{i_1 i_2 i_3},\\
&(b'') \quad \rho\pi'_{i_1 i_3 i_4} = \pm (\vartheta_{i_1}\vartheta_{i_2}/\Pi)^{2p-2} \cdot \pi_{i_2 i_3 i_4}\pi_{i_1 i_2 i_5} \cdots \pi_{i_1 i_2 i_{2p+2}},\\
&(c'') \quad \rho\pi'_{i_2 i_3 i_4} = \pm (\vartheta_{i_1}\vartheta_{i_2}/\Pi)^{2p-2} \cdot \pi_{i_1 i_3 i_4}\pi_{i_1 i_2 i_5} \cdots \pi_{i_1 i_2 i_{2p+2}}.
\end{aligned}$$

Thus linear spreads on p_{i_2} become spreads of order $2p-1$ on p_{i_1}, and vice versa; whence the involution is the element $I_{i_1 i_2}$ of $G_{2^{n+2}}$ $(n=2p-1)$. Since $I_{i_1 i_2}$ leaves w_p unaltered and on w_p is represented by $u' \equiv u + P_{i_1 i_2}$, then the symmetrical element $I = I_{i_1 \ldots i_{2p+2}}$ of $G_{2^{n+2}}$ leaves w_p unaltered and on w_p is represented by $u' \equiv u + P_{i_1 \ldots i_{2p+2}} \equiv u$. Hence w_p is the locus of fixed points of I and coincides with W_p. The corresponding proof for the case p odd can be formulated by a similar procedure. Hence

(60) *The equations* (42), (43) *furnish the parametric equations of the generalized Weddle manifold* W_p *in* S_{2p-1}. *If only those equations* (42), (43) *which omit the index* $2p+2$ *in* $\pi_{i_1 i_2 i_3}$ *are retained, they furnish the parametric equations of* W_p' *in* S_{2p-2}, *the projection of* W_p *from* p_{2p+2}. *On* W_p *and* W_p' *the equation of the involution* $I_{i_1 \ldots i_{2l}}$ *is, in parametric form,* $u' \equiv u + P_{i_1 \ldots i_{2l}}$.

16. Mapping of W_p upon the generalized hyperelliptic Kummer manifold K_p. We revert to the formulae (47) for the transcendental equations of the P-loci of P^{2p-1}_{2p+2} and find that

$$(61) \qquad \begin{aligned} &p \text{ even}: \quad \pi_{i_1 \ldots i_{2k+1}} \pi_{i_{2k+2} \ldots i_{2p+2}} = \vartheta^2_{i_1 \ldots i_{2k+1}} \cdot \Pi^{p-1}; \\ &p \text{ odd}: \quad \pi_{i_1 \ldots i_{2k+1}} \pi_{i_{2k+2} \ldots i_{2p+1}, 2p+2} = \vartheta^2_{i_1 \ldots i_{2k+1}, 2p+2} \cdot \Pi^{p-1}. \end{aligned}$$

On the left we have the pairs (30) whose products are found in the linear system Σ of (31a). On the right we have, to within the factor of proportionality Π^{p-1}, the squares of the 2^{2p} odd and even functions of the first order. Of these only 2^p are linearly independent and an independent set may be taken to be coördinates of a point in the linear space S_{2^p-1}. For variable u this point runs over the manifold K_p, whence

(62) *The equations* (61) *represent a mapping of* W_p *upon* K_p *by means of the linear system* Σ *of spreads of order* p *with* $(p-1)$*-fold points at* P^{2p-1}_{2p+2}. *The section of* W_p *by either member of a pair of* P*-loci is mapped upon the section of* K_p *by one of its* 2^{2p} *singular spaces.*

We inquire as to the behavior of the singular points of K_p with reference to W_p. As is known, these occur on K_p when u is a half period. With respect to W_p we observe that the F-locus of the p-th kind of P^{2p-1}_{2p+2}, $\pi^{(p)}_{i_1 \ldots i_p} = \pi^{(p)}_{i_{p+1} \ldots i_{2p+2}}$, which is the linear space S_{p-1} on $p_{i_1}, \cdots, p_{i_p}$, is cut by the system Σ in a system of p-ic spreads with $(p-1)$-fold points at $p_{i_1}, \cdots, p_{i_p}$. But such a spread in S_{p-1} is unique. Hence the entire space S_{p-1} is mapped upon a single point of K_p which we proceed to identify with a singular point.

Let the hyperelliptic curve $H_p{}^{p+2}$ satisfy the p further projective con-

ditions that $p+2$ of the branch points, $r_{i_{p+1}}, \cdots, r_{i_{2p+2}}$ are on a line L. Then the remaining p branch points are p flex-points at the p-fold point O, and the p-ad at O coincides with its superposed p-ad. Thus the parameters u of this p-ad must be a half period. When this situation arises, a number of the discriminant conditions are satisfied, and indeed, in many cases, are satisfied in more than one way.

Let the indices $j_1, j_2, \cdots$ be drawn from $i_{p+1}, \cdots, i_{2p+2}$. Of the two alternative forms (35) of the same discriminant condition we shall choose that which has more indices j than indices i $(i_1, \cdots, i_p)$. In

$$\delta(O^k j_1 \cdots j_{k+b+3} i_1 \cdots i_{k-b})^{k+1} = 0$$

this requires that $k+b+3 > k-b$. Also $1 \gtrless k+b+3 \gtrless p+2$ and $0 \gtrless k-b \gtrless p$. Hence

$$b > -3/2, \quad b \gtrless k, \quad b \gtrless p-k-1.$$

This permits of arranging the 2^{2p} distinct types of discriminant conditions in the order

$$\begin{aligned} &\text{(a)}: \quad b = -1;\ k = 0, 1, \cdots, p-1; \\ &\text{(b)}: \quad b = b;\ k = -b, b+1, \cdots, p-1-b \\ &\qquad\qquad [b = 0, 1, \cdots, (p-2)/2 \text{ or } (p-1)/2]. \end{aligned} \tag{63}$$

The line L must factor from the curve implied by the condition since

$$k+b+3 = (k+1)+(b+2) > k+1.$$

The remaining part must be a curve of order k with k-fold point at O and on $r_{i_1}, \cdots, r_{i_{k-b}}$. This part therefore consists of $k-b$ fixed lines on O and b variable lines on O. Hence if $b = -1$, the discriminant condition is not satisfied, whereas if $b > -1$ it is satisfied in $b+1$ linearly independent ways.

The point x on W_p which corresponds to this H_p^{p+2} with $r_{i_{p+1}}, \cdots, r_{i_{2p+2}}$ on L must (cf. (36)) lie on all the linear P-loci $\pi_{j_1 j_2 j_3}$ [for $b = 0$, $k = 0$ in (63b)], and therefore must be a point of the linear F-locus of the p-th kind $\pi^{(p)}{}_{i_1 \ldots i_p}$. To the discriminant condition $\delta(O^k j_1 \cdots j_{k+b+3} i_1 \cdots i_{k-b})^{k+1} = 0$ there correspond the conditions that x lies on $\pi_{j_1 \ldots j_{k+b+3} i_1 \ldots i_{k-b}}$ and on $\pi_{j_{k+b+4} \ldots j_{p+2} i_{k-b+1} \ldots i_p}$. The first of these two P-loci consists (cf. (17)) of the ∞^k linear spaces S_{2p-k-1} which are $(2p-k)$-secant to N^{2p-1} at $p_{j_{k+b+4}}, \cdots, p_{j_{p+2}} p_{i_{k-b+1}}, \cdots, p_{i_p}$ and at k variable points of N^{2p-1}. If, of these k variable points, $k-b$ are taken at $p_{i_1}, \cdots, p_{i_{k-b}}$, there must be ∞^b of the spaces S_{2p-k-1} on $\pi^{(p)}{}_{i_1 \ldots i_p}$; and thus $\pi^{(p)}$ is a $(b+1)$-fold locus on π. The second of the two P-loci consists of the ∞^{p-k-2} linear spaces S_{p+k} which are $(p+k+1)$-secant to N^{2p-1} at $p_{j_1}, \cdots, p_{j_{k+b+3}}, p_{i_1}, \cdots, p_{i_{k-b}}$ and at $p-k-2$

variable points of N^{2p-1}. None of these spaces S_{p+k} can contain the further $p+b-k$ points p_i on $\pi^{(p)}$ whence the P-locus does not contain $\pi^{(p)}$. Hence

(64) *The F-locus of the p-th kind,* $\pi^{(p)}{}_{i_1 \dots i_p} = \pi^{(p)}{}_{i_{p+1}, \dots, i_{2p+2}}$, *is a* $(b+1)$-*fold locus of the P-locus of the first kind* $\pi_{j_1 \dots j_{k+b+3} i_1 \dots i_{k-b}}$ (b *and* k *chosen as in* (63)), *and is not on the complementary P-locus. Thus it is a* $(b+1)$-*fold locus on the corresponding product of the system* (61). *When* x *on* W_p *is also on* $\pi^{(p)}$, *the* $p+2$ *branch points,* $r_{j_1}, \cdots, r_{j_{p+2}}$, *of* H_p^{p+2} *are on a line.*

We have yet to identify this F-locus, $\pi^{(p)}{}_{i_1 \dots i_p}$, with a *particular* half period. The following theorem (cf. [8] p. 460) states the behavior of the hyperelliptic odd and even thetas with respect to the half periods, when these are expressed in the basis notation.

(65) *If the two sets of indices* $i_1 \cdots i_{2l}$ *and* $j_1 j_2 \cdots$ *contain* $p+1 \pm 2r$ *indices which do not appear in both sets, then if* $r > 0$ *the theta function* $\vartheta_{j_1 j_2} \cdots$ *vanishes, together with its derivatives up to the order* $r-2$ *inclusive, for the half period* $P_{i_1 \dots i_{2l}}$.

The non-vanishing for $r = 0$ is normal, i. e., is common to all theta functions, as well as the vanishing of the functions for $r = 1$; but the further evanescences are characteristic of the hyperelliptic functions.

When p is even we have identified in (42) the P-locus $\pi_{j_1 \dots j_{k+b+3} i_1 \dots i_{k-b}}$ with the theta function $\vartheta_{j_1 \dots j_{k+b+3} i_1 \dots i_{k-b}}$. By using the rule (65) we find that this function does not vanish for the half period $P_{i_1 \dots i_p}$ if $b = -1$ but does vanish for $b = 0$, $b = 1$. For values $b > 1$ it vanishes together with its derivatives up to and including the order $b-1$. Hence $\pi^{(p)}{}_{i_1 \dots i_p}$ is to be identified with the half period $P_{i_1 \dots i_p}$. By applying the operations (27) and (60) we find that the half period $u = 0$ is to be identified with the F-locus, $\pi^{(p)}$, the locus of the $\infty^{p/2}$ $S_{(p-2)/2}$'s on $p/2$ variable points of N^{2p-1}.

When p is odd, the P-locus $\pi_{j_1 \dots j_{k+b+3} i_1 \dots i_{k-b}}$ is identified with $\vartheta_{j_1 \dots j_{k+b+3} i_1 \dots i_{k-b}, 2p+2}$. If then the half period $P_{i_1 \dots i_p, 2p+2}$ is identified with the F-locus, $\pi^{(p)}{}_{i_1 \dots i_p}$, the behavior of the P-loci and the theta functions with respect to respectively the F-locus and the half period coincides with that just described. This ascribes to the improper half period, $u = 0$, the F-locus. $\pi^{(p)}{}_{2p+2}$, the locus of $\infty^{(p-1)/2}$ $S_{(p-1)/2}$'s on p_{2p+2} and $(p-1)/2$ variable points of N^{2p-1}. Hence

(66) *The* 2^{2p} *half periods determine on* W_p *the* 2^{2p} *F-loci of the p-th kind as follows:*

$$p \text{ even}: \quad \pi^{(p)}{}_{i_1 \dots i_{2k+2-p}} = \pi^{(p)}{}_{i_{2k+3-p} \dots i_{2p+2}} \quad : \quad P_{i_1 \dots i_{2k+2-p}};$$

$$p \text{ odd}: \quad \pi^{(p)}{}_{i_1 \dots i_{2k+2-p}} = \pi^{(p)}{}_{i_{2k+3-p} \dots i_{2p+2}} \quad : \quad P_{i_1 \dots i_{2k+2-p}, 2p+2}.$$

These F-loci are mapped as in (62) *into the* 2^{2p} *singular points of* K_p.

The mapping (62) furnishes a purely algebraic definition of the Kummer manifold K_p since W_p itself is the locus of fixed points of a Cremona involution defined by P_{2p+2}^{2p-1}. This algebraic definition suggests properties of K_p which may not readily appear from its transcendental definition. We examine in (83) one such situation for $p = 3$.

§ 4

The Spread W_3' in S_4.

17. Theta relations in the hyperelliptic case, $p = 3$. In the present case the functions are hyperelliptic when $\vartheta(0) = c = c_{12\ldots8} = 0$. The three-term relations are obtained from those four-term relations connecting the four products of four pairs of odd functions in a Steiner complex [[1] p. 168, (3)]. With reference to the basis notation there are two types of Steiner complexes corresponding to the two types of half periods, P_{78} and P_{5678}. In the second-type the four pairs can be selected in two distinct ways so that three types of four-term relations exist which connect the functions indicated:

$$\begin{array}{lllll} \text{A:} & \vartheta_{17}\vartheta_{18}, & \vartheta_{27}\vartheta_{28}, & \vartheta_{37}\vartheta_{38}, & \vartheta_{47}\vartheta_{48}; \\ \text{B:} & \vartheta_{23}\vartheta_{14}, & \vartheta_{31}\vartheta_{24}, & \vartheta_{12}\vartheta_{34}, & \vartheta_{56}\vartheta_{78}; \\ \text{C:} & \vartheta_{31}\vartheta_{24}, & \vartheta_{12}\vartheta_{34}, & \vartheta_{56}\vartheta_{78}, & \vartheta_{57}\vartheta_{68}. \end{array}$$

The coefficients of the functions in these relations are, to within a $\pm$ sign:

$$\begin{array}{lllll} \text{A}': & c_{2347}c_{2348}, & c_{3147}c_{3148}, & c_{1247}c_{1248}, & c_{1237}c_{1238}; \\ \text{B}': & c_{2378}c_{1478}, & c_{3178}c_{2478}, & c_{1278}c_{3478}, & cc_{1234}; \\ \text{C}': & c_{1258}c_{3458}, & c_{3158}c_{2458}, & c_{5714}c_{6814}, & c_{5614}c_{7814}. \end{array}$$

When $c = 0$, Thomae's values (52) of the c_{ijkl} enable us to express these coefficients in terms of the roots of the fundamental octavic as follows:

$$\begin{array}{ll} \text{A}'': & [\{234\}\{156\}]^{1/2},\ [\{314\}\{256\}]^{1/2},\ [\{124\}\{356\}]^{1/2},\ [\{123\}\{456\}]^{1/2}; \\ \text{B}'': & [(23)(14)]^{1/2},\ [(31)(24)]^{1/2},\ [(12)(34)]^{1/2},\ 0; \\ \text{C}'': & [(12)(34)(58)(67)]^{1/2},\ [(31)(24)(58)(67)]^{1/2}, \\ & \quad [(14)(23)(57)(68)]^{1/2},\ [(14)(23)(56)(78)]^{1/2}. \end{array}$$

By adding a half period to u in these relations other relations of different form are obtained. We are concerned primarily with the three-term relations obtained from B, B''. The three terms are symmetric with respect to a division of the indices, 1234, 5678; and the addition of P_{1234} leaves each

term invariant. To secure typical results it is thus sufficient to add the half periods, P_{78}, P_{48}, P_{34}, P_{3478}. We secure the following five sets of three functions, each in a relation with coefficients B″:

$$\begin{array}{lccc} \text{B1:} & \vartheta_{23}\vartheta_{14}, & \vartheta_{31}\vartheta_{24}, & \vartheta_{12}\vartheta_{34}; \\ \text{B2:} & \vartheta_{2378}\vartheta_{1478}, & \vartheta_{3178}\vartheta_{2478}, & \vartheta_{1278}\vartheta_{3478}; \\ \text{B3:} & \vartheta_{2348}\vartheta_{18}, & \vartheta_{3148}\vartheta_{28}, & \vartheta_{1248}\vartheta_{38}; \\ \text{B4:} & \vartheta_{24}\vartheta_{13}, & \vartheta_{14}\vartheta_{23}, & \vartheta_{1234}\vartheta; \\ \text{B5:} & \vartheta_{2478}\vartheta_{3178}, & \vartheta_{1478}\vartheta_{2378}, & \vartheta_{56}\vartheta_{78}. \end{array}$$

Since, for the odd value of p, the index $2p+2=8$ is isolated in the application we are about to make, the relations just mentioned yield others as follows:

$$\begin{array}{lccll} \text{B1b:} & \vartheta_{23}\vartheta_{18}, & \vartheta_{31}\vartheta_{28}, & \vartheta_{12}\vartheta_{38} & : (48); \\ \text{B2b:} & \vartheta_{1256}\vartheta_{3456}, & \vartheta_{1257}\vartheta_{3457}, & \vartheta_{1267}\vartheta_{3467} & : (17)(826354); \\ \text{B3b:} & \vartheta_{1567}\vartheta_{14}, & \vartheta_{2567}\vartheta_{24}, & \vartheta_{3567}\vartheta_{34} & : (48); \\ \text{B2c:} & \vartheta_{2345}\vartheta_{15}, & \vartheta_{1345}\vartheta_{25}, & \vartheta_{1245}\vartheta_{35} & : (58); \\ \text{B3d:} & \vartheta_{1234}\vartheta_{18}, & \vartheta_{2567}\vartheta_{12}, & \vartheta_{3567}\vartheta_{13} & : (81); \\ \text{B4b:} & \vartheta_{56}\vartheta_{78}, & \vartheta_{57}\vartheta_{68}, & \vartheta_{1234}\vartheta & : (48)(27)(16)(35); \\ \text{B5b:} & \vartheta_{1256}\vartheta_{3456}, & \vartheta_{1257}\vartheta_{3457}, & \vartheta_{12}\vartheta_{34} & : (71825463). \end{array}$$

The substitution (in cycle form) on the right of each is that which converts *Bi* into the given form; and which therefore must be applied also to the coefficients *B″* to obtain the coefficients of the given terms.

The three-term relations have been derived afresh to show their place in the relations among the non-hyperelliptic functions. Thy might have been obtained directly from (44b).

18. The theta relations of 17 as projective relations in S_4. The P-loci of the set $P_7{}^4$ in S_4, π_{i_1}, $\pi_{i_1 i_2 i_3}$, $\pi_{i_1 \cdots i_5}$, and $\pi_{i_1 \cdots i_7}$ $(i_j = 1, \cdots, 7)$ are defined in (47) in terms of the thetas:

$$(67a) \qquad \pi_{i_1 \cdots i_{2k+1}} = \vartheta_{8i_1 \cdots i_{2k+1}} \cdot \Pi^k / \vartheta_{8i_1} \cdots \vartheta_{8i_{2k+1}} \qquad (k = 0, \cdots, 3),$$
$$\Pi = \vartheta_{81}\vartheta_{82} \cdots \vartheta_{87}.$$

Three-term relations among products of these P-loci are then derived from the theta relations. We have noticed already in (59), (57) an example of a projective relation in S_4 which is obtained from a theta relation. We now extend the definition of $G_{i_1 i_2 i_3}$ in (59) to cover all the P-loci of $P_7{}^4$ by setting

$$(67b) \qquad G_{i_1 \cdots i_{2k+1}} = \pi_{i_1 \cdots i_{2k+1}} \cdot \{i_1 \cdots i_{2k+1}\}^{1/2}.$$

The factors, $\{i_1 \cdots i_{2k+1}\}$, are defined in (49) in terms of the differences

(ij) of the roots t_i, t_j $(i, j = 1, \cdots, 7)$ of the fundamental octavic. These differences are proportional to the quinary determinants formed from the coördinates of the points of P_7^4 as in (50); specifically

$$(67c) \qquad (i_1 \cdots i_5) = \lambda \cdot \epsilon_{i_1 \ldots i_5 i_6 i_7} (i_6 i_7).$$

In order that a relation involving these differences may be interpreted as a projective relation, it is necessary that a certain number (the same for each term) of the differences be interpreted as quinary determinants so that the terms may be homogeneous in each point of P_7^4; and that the products of the remaining differences in each term be homogeneous in each of the roots $t_1, \cdots, t_8$ so that the ratios of the products are invariants, rational or irrational, of the octavic. An example is the transition from (57) to the preceding purely projective relation.

We prove that the functions G satisfy the same system of relations as the projective expressions in (67d) and therefore may be identified with them.

$$(67d) \quad \begin{aligned} G_{123} &= \pm (4567x) \\ G_{1\ldots5} &= \pm [(23)(14)(2467x)(1367x) - (13)(24)(1467x)(2367x)], \\ G_{1\ldots7} &= \pm [(23)(14) G_{13567} G_{24567} - (13)(24) G_{23567} G_{14567}]. \end{aligned}$$

Thus the respective degrees of G_{123}, $G_{1\ldots5}$, and $G_{1\ldots7}$ in the coördinates of P_7^4 are

$$(0\ 0\ 0\ \ 1\ 1\ 1\ 1), \qquad (2\ 2\ 2\ 2\ 2\ 4\ 4), \qquad (6\ 6\ 6\ 6\ 6\ 6\ 6).$$

Multiply B3 in **17** by $\Pi/\vartheta_{18} \cdots \vartheta_{48}$ to introduce the functions π in (67a); with coefficients B'', then insert the values G from (67b); and multiply by $\{4;\ 123\}^{1/2}$ to remove the denominators. The resulting relation is

$$(14) G_{234} \pm (24) G_{314} \pm (34) G_{124} \equiv 0.$$

With the differences replaced by quinary determinants this is the projective relation satisfied by S_3's on p_5, p_6, p_7 to p_1, p_2, p_3 respectively, and justifies the first formula of (67d).

Again we multiply in B5 by $\Pi^2/\vartheta_{18} \cdots \vartheta_{48} \vartheta^2_{78}$ to introduce the functions π, and after inserting the coefficients B'' and the functions G, multiply by $[\{12347\}/(12)(34)]^{1/2}$ to obtain

$$(23)(14) G_{247} G_{317} \pm (24)(13) G_{147} G_{237} \pm G_{12347} \equiv 0.$$

On interchanging 7 and 5, this is the second formula (67d). The determination of the relative sign of the first two terms given in (67d) is based on the fact that the third term, $G_{12347} = \rho \pi_{12347}$, must represent a quadric with nodes at p_5, p_6 and on $p_1, \cdots, p_4, p_7$.

Finally on multiplying B4 by $\Pi^3/\vartheta_{58}\vartheta_{68}\vartheta^2_{78}$, and by

$$[\{567\}\{1234567\}/(12)(34)]^{1/2},$$

we convert it into the projective relation,

$$(23)(14)G_{13567}G_{24567} - (13)(24)G_{23567}G_{14567} = \pm G_{567}G_{1\ldots7},$$

which is the third formula of (67d). We verify in a moment that

$$\begin{aligned}(67e)\quad G_{1234567} = \pm\,[&(14)(46)(63)(32)(25)(51)(1342x)(1365x)(4265x)\\ &(14)(42)(23)(36)(65)(51)(1346x)(1325x)(4625x)\\ &(14)(46)(65)(52)(23)(31)(1542x)(1563x)(4263x)\\ &(13)(36)(64)(42)(25)(51)(1432x)(1465x)(3265x)\\ &(13)(36)(65)(52)(24)(41)(1532x)(1564x)(3264x)\\ &(13)(32)(24)(46)(65)(51)(1436x)(1425x)(3625x)].\end{aligned}$$

We complete the examination of the relations, $B1, \cdots, B5$, by multiplying $B1$, $B2$ respectively by

$$[\{567;\ 1234\}\{567\}^2]^{1/2}\,\Pi^3/\vartheta_{58}\vartheta_{68}\vartheta_{78},$$
$$[\{7;\ 1234\}]^{1/2}\,\Pi^2/\vartheta_{18}\cdots\vartheta_{48}\vartheta^2_{78}$$

to reduce them to

$$\begin{aligned}(67f)\qquad &G_{14567}G_{23567} \pm G_{24567}G_{31567} \pm G_{34567}G_{12567} \equiv 0,\\ &G_{237}G_{147} \pm G_{317}G_{247} \pm G_{127}G_{347} \equiv 0.\end{aligned}$$

The second of these two is the identity which arises when the determinants in the ordinary binary identity, $(14)(23)+(24)(31)+(34)(12)=0$, are expanded into the quinary domain by the insertion of $56x$ in each determinant according to the Clebsch principle of transference. The first is the form which the second takes when the Cremona involution I_{56} is applied to it. A more direct proof is not immediate so that this identity is an example of the usefulness of the thetas in a projective study of $P_7{}^4$ and related loci.

A similar example is furnished by B3c which yields

$$(67g)\qquad (14)G_{167}G_{23467} \pm (24)G_{267}G_{31467} \pm (34)G_{367}G_{12467} \equiv 0.$$

The Cremona involution I_{46} applied to the second relation (67f) will produce an identity of this character.

The transcription of the relations B1b, $\cdots$, B5b and B3d is not so direct and will be deferred until more information concerning W_3' is gained (cf. **19**). We close this section with an examination of $G_{1\ldots7}$ in (67e), the interesting locus of the bisecants of the curve N^4 on $P_7{}^4$ (cf. **4**, table). If in (67d) we take the quadrics in the form,

$$G_{13567} = (2413x)(2456x)(15)(36) - (2415x)(2436x)(13)(56),$$
$$G_{24567} = (1324x)(1356x)(25)(46) - (1325x)(1346x)(24)(56),$$
$$G_{23567} = (1423x)(1456x)(25)(36) - (1425x)(1436x)(23)(56),$$
$$G_{14567} = (2314x)(2356x)(15)(46) - (2315x)(2346x)(14)(56),$$

then the combination, $(23)(14)G_{13567}G_{24567} - (13)(24)G_{23567}G_{14567}$ contains the factor, $G_{567} = (1234x)$. Indeed the only two terms of the combination, which do not contain this factor explicitly, cancel each other. The complementary factor is the bracket in (67e). Each of the terms in $G_{1\ldots7}$ involves one of the cyclic arrangements:

$$146325, \quad 142365, \quad 146523, \quad 136425, \quad 136524, \quad 132465.$$

For each cycle the pairs of diagonals of the corresponding hexagon determine the quinary determinants. The six cycles may be described as the *six ways in which* 126 *can be inserted between* 345 *or vice versa.* Thus there must be ten formally different equations (67e) of $G_{1\ldots7}$; or $10.7 = 70$ if also the exceptional position of p_7 be considered.

It is evident from the given form that the cubic spread $G_{1\ldots7}$ has nodes at $p_1, \cdots, p_6$. That it has a double point at p_7 is also easy to see. For, the second polar of p_7 is obtained by setting $x = p_7$ in each pair of terms in each of the six products of three quinary determinants. Then any determinant, e. g. $(1234x)$, appears twice with coefficients which cancel when (13567) is replaced by (24), etc. On account of the symmetry it is sufficient to verify this for $(1234x)$ alone. Since p_7 does not appear explicitly in the equation the locus must have a node at every point $t = t_7$ on N^4, and therefore must be the locus of bisecants of N^4.

19. Geometric properties of W_3'. We have seen [cf. (66)] that the F-loci of the third kind of $P_8{}^5$ in S_5 are contained on W_3. When W_3 is projected from p_8 upon W_3' in S_4 these F-loci of the third kind of $P_8{}^5$ are projected into the F-loci of the second kind of $P_7{}^4$ (cf. **4**, table). Hence W_3' must contain these F-loci of the second kind of $P_7{}^4$ for the values of the half periods as follows:

$$u = 0 \quad : \quad \pi^{(2)} = N^4;$$
$$u = P_{i_1 i_2} \quad : \quad \pi^{(2)}{}_{i_1 i_2} = S_1(i_1 i_2)^1;$$
$$u = P_{i_1 i_2 i_3 i_4} \quad : \quad \pi^{(2)}{}_{i_1 i_2 i_3 i_4} = S_2(i_5 i_6 i_7)^1;$$
$$u = P_{i_7 8} \quad : \quad \pi^{(2)}{}_{i_1 \ldots i_7} = C^3{}_{i_7} = M_2(i_1 \cdots i_6 i_7{}^3)^3.$$

Since W_3' is invariant under the G_{64} determined by $P_7{}^4$ it must be a spread of order $5k$ with $3k$-fold points at $P_7{}^4$. Moreover since the same pro-

jective set P_7^4 can be obtained by the projection of a P_8^5 in S_5 which consists of P_7^5 and any point t on the N^5 determined by P_7^5 and p_8, the equation of W_3' in S_4 can involve $t = t_8$ only in the coefficients. Hence W_3' will be a member of a system (∞^1) determined by P_7^4. We shall find that this system is a pencil.

The projective relations thus far obtained from the theta relations have been identities among the P-loci which were valid throughout the space. These could lead to no definition of W_3'. We consider then the further set B1b, $\cdots$, B5b. In order to handle the functions $\vartheta^2_{18}, \cdots, \vartheta^2_{78}$, which do not disappear as in the earlier cases, we define two new sets of seven functions, $H_1, \cdots, H_7$; $H_1', \cdots, H_7'$, as follows:

$$(68a)\qquad \begin{aligned} H_1 &= \vartheta^2_{18} \cdot w\Pi^2 \cdot [\{234567\}(18)]^{1/2}, \\ H_1' &= [1/\vartheta^2_{18}] \cdot w\Pi^4 \cdot [\{1234567\}\{1;\ 234567\}(18)]^{1/2}, \end{aligned}$$

where w is a function to be determined later. Hence

$$(68b)\qquad \begin{aligned} H_1H_1' = (18) \cdot \Pi^6 w^2 \cdot \{1234567\}, \cdots, H_7H_7' &= (78) \cdot \Pi^6 w^2 \cdot \{1234567\} \\ (23)H_1H_1' + (31)H_2H_2' + (12)H_3H_3' &\equiv 0. \end{aligned}$$

In terms of the functions H, H' the later theta relations transcribe as follows:

$$(68c)\qquad \begin{aligned} &\text{B1b:}\ G_{14567}H_1 \pm G_{24567}H_2 \pm G_{34567}H_3 \equiv 0; \\ &\text{B2b:}\ G_{347}G_{127}H_7 \pm G_{346}G_{126}H_6 \pm G_{345}G_{125}H_5 \equiv 0; \\ &\text{B3b:}\ G_{234}G_{23567}H_1' \pm G_{314}G_{31567}H_2' \pm G_{124}G_{12567}H_3' \equiv 0; \\ &\text{B4b:}\ (57)G_{12347}H_6' \pm (56)G_{12346}H_7' \pm G_{567}G_{1\ldots7}H_5 \equiv 0; \\ &\text{B5b:}\ (57)G_{127}G_{347}H_6' \pm (56)G_{126}G_{346}H_7' \pm G_{12567}G_{34567}H_5 \equiv 0. \end{aligned}$$

If B4b and B5b are multiplied by H_6H_7, they reduce by using (68b) to

$$(68d)\qquad \begin{aligned} &\text{B4b':}\ (57)(68)G_{12347}H_7 \pm (56)(78)G_{12346}H_6 \\ &\qquad \pm G_{567}G_{1\ldots7}H_5H_6H_7/\Pi^6w^2\{1\cdots7\} \equiv 0, \\ &\text{B5b':}\ (57)(68)G_{127}G_{347}H_7 \pm (56)(78)G_{126}G_{346}H_6 \\ &\qquad \pm G_{12567}G_{34567}H_5H_6H_7/\Pi^6w^2\{1\cdots7\} \equiv 0. \end{aligned}$$

We write the identity B2b for the three combinations, 12, 34; 13, 42; and 14, 23, using the projective form except in the unknown H_5, H_6, H_7. Thus

$$\begin{aligned} &\pm(1256x)(3456x)H_7 \pm (1257x)(3457x)H_6 + (1267x)(3467x)H_5 \equiv 0, \\ &\pm(1356x)(4256x)H_7 \pm (1357x)(4257x)H_6 + (1367x)(4267x)H_5 \equiv 0, \\ &\pm(1456x)(2356x)H_7 \pm (1457x)(2357x)H_6 + (1467x)(2367x)H_5 \equiv 0. \end{aligned}$$

If these be added the terms in H_5 vanish due to the second identity (67f). Since H_6 and H_7 with significant factors ϑ^2_{68}, ϑ^2_{78} could not remain, the terms containing them must also vanish due to the same identity, whence the ambiguous signs are alike in each column. Any two of the three identities then serve to determine the ratios of H_5, H_6, H_7. We shall set

$$(68e) \qquad H_7 = \pm \begin{vmatrix} (1257x)(3457x) & (1267x)(3467x) \\ (1357x)(4257x) & (1367x)(4267x) \end{vmatrix} = \begin{vmatrix} G_{346}G_{126} & G_{345}G_{125} \\ G_{426}G_{136} & G_{425}G_{135} \end{vmatrix} .$$

This expression for H_7 determines the function w in (68a). For, if the quinary determinants are replaced from (67a, b, d), H_7 takes the form

$$(68f) \qquad H_7 = \pm \Pi^2 \cdot \vartheta^2_{78} \cdot [\{123456\}(78)]^{1/2} \cdot \frac{\vartheta_{3468}\vartheta_{1268}\vartheta_{2458}\vartheta_{1358} \pm \vartheta_{3458}\vartheta_{1258}\vartheta_{2468}\vartheta_{1368}}{[(14)(23)(56)(78)]^{1/2}}$$

On comparing this with (68a) we find that

$$(68g) \qquad w = (\vartheta_{3468}\vartheta_{1268}\vartheta_{2458}\vartheta_{1358} \pm \vartheta_{3458}\vartheta_{1258}\vartheta_{2468}\vartheta_{1368})/[(14)(23)(56)(78)]^{1/2}.$$

We recognize at once from (68e) that H_7 is the locus of points x such that $p_1, \cdots, p_6$ are projected from the line p_7x into six points on a conic; or, it is the locus of points x which are projected from p_7 into the points of a Weddle quartic surface with nodes at the projections of $p_1, \cdots, p_6$. It is thus a Weddle quartic cone with 4-fold point at p_7 and double lines from p_7 to $p_1, \cdots, p_6$. It contains the remaining F-loci of the second kind and dimension one simply [i. e., N^4 and the 15 lines $S_1(i_1i_2)^1$ $(i_1, i_2 = 1, \cdots, 6)$]; and also contains F-loci of the second kind and dimension two; namely $C_7{}^3$, and the 15 planes $S_2(7i_1i_2)^1$.

The identity B1b brings out the significance of the factor w in the definition of the Weddle cones H. The two sextic spreads $G_{14567}H_1$ and $G_{24567}H_2$ lie in a pencil with a basic 2-way of order 36. The quadrics G_{14567} and G_{24567} meet in $C_3{}^3$ and $S_2(123)^1$; G_{14567} and H_2 meet in $C_2{}^3$ and $S_2(231)^1$, $S_2(234)^1, \cdots, S_2(237)^1$; G_{24567} and H_1 meet in $C_1{}^3$ and $S_2(132)^1$, $S_2(134)^1$, $\cdots, S_2(137)^1$; and H_1 and H_2 meet in $S_2(123)^1$, $S_2(124)^1, \cdots, S_2(127)^1$ and a $V_2{}^{11}$. Since the complete intersection of H_1 and H_2 is a 16-ic with 8-fold points at p_1, p_2 and 4-fold points at $p_3, \cdots, p_7$, $V_2{}^{11}$ must have triple points at $P_7{}^4$. Since $G_{34567}H_3$ is a member of this pencil, $V_2{}^{11}$ must also lie on H_3, and thus must lie on each of the seven Weddle cones. Also H_1, H_2 contain N^4 and the line $S_1(34)^1$. These not being on the 5 planes common to H_1, H_2 must be on $V_2{}^{11}$. On account of the symmetry of $V_2{}^{11}$ with respect to the seven cones we may state that

(69) *The seven Weddle quartic cones H_i with respectively vertices at p_i*

and double lines p_ip_j meet in a surface V_2^{11} with triple points at P_7^4 and containing N^4 and the 21 lines $S_1(ij)$[1]. The surface V_2^{11} is invariant under the Cremona G_{64} defined by P_7^4. The equation of this surface on W_3' is $w=0$ (cf. 68g).

For, the Weddle cones, H_6 and H_7, are each invariant under the involutions generated by $I_{i_1i_2}$ $(i_1, i_2 = 1, \cdots, 5)$; whence V_2^{11}, their common part except for F-loci, is also invariant under these involutions. That V_2^{11} is on W_3' is a consequence of the equation of W_3' we are about to derive.

If in (68d) the third term of the two identities is eliminated, we find that

$$(70)\qquad \begin{aligned} W_3' = {} & (57)(68)\cdot[G_{12567}G_{34567}G_{12347} - G_{567}G_{127}G_{347}G_{1\ldots7}]\,H_7 \\ & \pm(56)(78)\cdot[G_{12567}G_{34567}G_{12346} - G_{567}G_{126}G_{346}G_{1\ldots7}]\,H_6 = 0. \end{aligned}$$

This relation (70) is an identity in u but it is not a projective identity. For, the S_3, $G_{567}=0$, is a factor of the two of the terms but not a factor of the other two for variable t_8. We observe that each of the brackets is of degree 10 in each point p of P_7^4 as well as of degree 10 in x. Hence the factors (57)(68), (56)(78) are to be interpreted in the binary domain as terms of a double ratio. Thus W_3' appears, as observed earlier, as a member of a pencil (for variable t_8) defined by P_7^4.

The alternative forms of the function w of (68g) are 105 in number. These arise from the 15 choices of the form of H_7 in (68e) which depend upon the division of the six indices into pairs (14)(23)(56); and from the 7 choices of the index 7.

The relations B3b play the same role in the determination of the functions H' as the relations B2b in the determination of the H's. Let B3b be written three times for the cyclic permutation of 4, 5, 6. The coexistence of the resulting three identities is a consequence of the identities (67g) with multipliers (47), (57), (67). Any two of the identities serve to determine the ratios of H_1', H_2', H_3'. If we select the first two, an extraneous constant factor (67) appears, so that we set

$$(71a)\qquad (67)\cdot H_1' = \pm \begin{vmatrix} G_{134}G_{13567} & G_{124}G_{12567} \\ G_{135}G_{13467} & G_{125}G_{12467} \end{vmatrix}.$$

That this projective form of H_1' coincides with the definition of H_1' in (68a) appears when the spreads G are expressed in terms of the functions by means of (67a, b). The expression is

$$\begin{aligned} (71b)\quad (67)\cdot H_1' = {} & \pm(67)\cdot[\Pi^4/\vartheta^2_{18}]\cdot[\{1\cdots7\}\{1;\,2\cdots7\}(18)]^{1/2} \\ & \cdot(\vartheta_{24}\vartheta_{35}\vartheta_{1348}\vartheta_{1258} \pm \vartheta_{34}\vartheta_{25}\vartheta_{1248}\vartheta_{1358})/[(18)(23)(45)(67)]^{1/2}. \end{aligned}$$

This value of H_1' coincides with that in (68a) except for the form of the function w. By equating two alternative forms of w in (68g),

$$(\vartheta_{3568}\vartheta_{1258}\vartheta_{1348}\vartheta_{2468} \pm \vartheta_{1358}\vartheta_{2658}\vartheta_{3468}\vartheta_{1248})/[(16)(23)(45)(78)]^{1/2}$$
$$= \pm(\vartheta_{3578}\vartheta_{2478}\vartheta_{3468}\vartheta_{2568} \pm \vartheta_{2578}\vartheta_{3478}\vartheta_{2468}\vartheta_{3568})/[(18)(23)(45)(67)]^{1/2},$$

we have an identity which, when u is replaced $u + P_{16}$, expresses the function in (71b) as one of the alternative forms of w. Thus (71a) is a projective definition of H_1'. According to it the degrees of H_1' in the points $p_1, \cdots, p_7$ are (3, 6, 6, 6, 6, 6, 6).

Other forms of H_7' occur; e. g., the coefficient of H_7 in (70). We find, again by using (67a, b), that this coefficient is

$$(12)(34)(56) \cdot [\Pi^4/\vartheta^2_{78}] \cdot [\{1 \cdots 7\}\{7; 1 \cdots 6\}(78)]^{1/2}.$$
$$\cdot (\vartheta_{34}\vartheta_{12}\vartheta_{56}\vartheta_{78} - \vartheta_{5678}\vartheta_{1278}\vartheta_{3478}\vartheta)/[(12)(34)(56)(78)]^{1/2}.$$

By the same argument this is $(12)(34)(56) \cdot H_7'$ whence

$$(71c) \qquad (12)(34)(56) \cdot H_7' = \begin{vmatrix} G_{12347} & G_{567}G_{1,\ldots 7} \\ G_{127}G_{347} & G_{12567}G_{34567} \end{vmatrix} .$$

On replacing the coefficients of H_7 and H_6 in (70) by these values we find that

$$(72) \quad W_3' = (12)(34) \cdot [(57)(68) \cdot (56)H_7H_7' \pm (56)(78) \cdot (57)H_6H_6'].$$

We observe that both terms, $(56)H_7H_7'$ and $(57)H_6H_6'$, have degrees (9, 9, 9, 9, 8, 8, 8); and the factor (12)(34) raises all of these degrees to 10.

The spread, $H_7' = 0$, is a sextic spread with four-fold points at $p_1, \cdots, p_6$ and double point at p_7, which is on $V_2{}^{11}$. According to its transcendental definition (68a), these properties are sufficient to define it. The Weddle quartic cone H_7 is invariant under the G_{32} whose involutions $I_{i_1 \ldots i_{12}}$ do not contain the index 7. It has therefore but one conjugate under G_{64}. The image of H_7 under I_{17} is a sextic spread with double point at p_7, 4-fold points at $p_1, \cdots, p_6$, and on $V_2{}^{11}$ which is invariant under I_{17} [cf. (69)]. This image must coincide with H_7'. Hence each of the seven products H_iH_i' is invariant under G_{64}. According to (68b) these seven products are in a pencil. We observe that when, in (72), $t_8 = t = t_7$, W_3' reduces to H_7H_7'. Hence

(73) *Each of the seven Weddle cones H_i is invariant under a G_{32} in G_{64}, and is conjugate under G_{64} to a sextic spread H_i' with node at p_i and 4-fold points at the other 6 points p. The seven products H_iH_i' are in a pencil with parameters $t = t_i$. The spread W_3' is in this pencil with parameter*

$t = t_8$. *The pencil is that of order* 10, *which contains* V_2^{11} *as a double locus and the F-loci of the second kind of* P_7^4 *as a simple locus.*

The base of the pencil, a surface of order 100, consists of V_2^{11} counted four times, of the seven cubic cones C_i^3, and of the 35 planes $S_2(ijk)^1$. We say that the pencil contains the one-dimensional F-loci as *simple* loci if it contains them to a multiplicity one greater than they must occur on spreads of order 10 with 6-fold points at P_7^4. Thus $S_1(12)^1$ must be a double line of spreads of such a pencil. It is however a triple line on the pencil (70), and a like situation with respect to N^4 and the 21 lines $S_1(ij)$ must exist.

We give finally a determination of sign for the spread W_3'.

(74) *If the signs of* H_1, H_2, H_3 *and* H_1', H_2', H_3' *are so chosen that the identity* (68b) *reads:*

$$(23)H_1H_1' + (31)H_2H_2' + (12)H_3H_3' \equiv 0,$$

then the equation of W_3' *can be written as*

$$\left|\left| \begin{matrix} (23)H_1H_1' & (31)H_2H_2' & (12)H_3H_3' \\ (23)(18) & (31)(28) & (12)(38) \end{matrix} \right|\right| = 0.$$

For, if $x_0 + x_1 + x_2 = 0$ and $y_0 + y_1 + y_2 = 0$, then $x_1y_2 - x_2y_1 = x_2y_0 - x_0y_2 = x_0y_1 - x_1y_0$. Only with this disposition of the $\pm$ signs could the three forms of W_3', as given in (72), coexist.

20. The double surface V_2^{11} of W_3'. Let L be a generic line of the Weddle quartic cone H_7 with vertex at p_7. This is a fixed line of the involution I_{123456} whose parametric equation on W_3' and V_2^{11} is $u' \equiv u + P_{78}$. The line L meets H_6 in two points outside p_7 which are points on V_2^{11} and double points on W_3', whence L meets V_2^{11} and W_3' only at these two points outside p_7. These points must be interchanged by $I_{1\ldots6}$, since $u' \equiv u + P_{78}$ has only 32 fixed points on W_3' which are determined by the quarter periods, $\pm P_{78}/2$. Since each point of V_2^{11} is on one line L, there follows that:

(75) *On* V_2^{11}, *invariant under* G_{64}, *the elements of* G_{64} *are generated by combining the involutions* I_{i8} $(i = 1, \cdots, 7)$ *which respectively are the projections of* V_2^{11} *into itself from* p_i. *Under successive projections from the triple points a point* u *of* V_2^{11} *gives rise to a closed set of* 64 *points conjugate under* G_{64}. *The surface* V_2^{11} *is projected from any one of its triple points, say* p_7, *into a doubly covered Weddle quartic surface with fundamental sextic* $t_1, \cdots, t_6$; *and the conjugate sets of* 64 *points on* V_2^{11} *are projected into sets of* 32 *points on the Weddle of the type described by Baker*[11] (*cf. also Hudson,*[12] p. 169).

This V_2^{11} is a particular case of a surface which occurs in connection with every set P^n_{n+3} $(n \geqq 3)$ in S_n. The rational curve of order $n+1$ in S_n, R^{n+1}, with a node can be put into the unique canonical form,

$$x_0 = (t_0^{n+1} + t_1^{n+1})/2, \quad x_i = t_0^{n+1-i} t_1^{\,i} \qquad (i = 1, \cdots, n).$$

It has therefore $n(n+2)$ constants. On P^n_{n+3} there are ∞^3 R^{n+1}'s with a node, since $n(n+2) - (n+3)(n-1) = 3$. If x is the node of such an R^{n+1}, the R^{n+1} is projected from x into an N'^{n-1} on the projected set P'_{n+3} in S_{n-1}. The cone of projection, K^{n-1}, has an $(n-1)$-fold point at x. Since N'^{n-1} is on $(n-1)(n-2)/2$ independent quadrics, K^{n-1} is contained on a system Σ' $(\infty^{-1+(n-1)(n-2)/2})$ of quadrics in S_n with a node at x. The system Σ $(\infty^{(n^2+n-8)/2})$ of quadrics in S_n on P^n_{n+3} and x is made up of Σ', and a system Σ'' (∞^{2n-5}) which does not contain K^{n-1}. The system Σ'' cuts K^{n-1} in a linear system of curves L'', of dimension $2n-5$, order $2n-2$, and with an $(n-1)$-fold point at x. The system L'' contains R^{n+1}'s with node at x together with $n-3$ arbitrary lines of K^{n-1}. It is two conditions on Σ'' to contain such a line whence there are ∞^1 R^{n+1}'s on K^{n-1}, which have a node at x and pass through P^n_{n+3}. Thus the locus of the node x of the ∞^3 nodal R^{n+1}'s on P^n_{n+3} is a surface. Hence

(76) *Given P^n_{n+3} there exists a surface $V_2^{(n)}$, the locus of a point x:* (a) *from which P^n_{n+3} projects into a set P'^n_{n+3} in S_{n-1} on a norm-curve N^{n-1};* (b) *which is the $(n-1)$-fold point of a cone K^{n-1} of order $n-1$ on P^n_{n+3};* *and* (c) *which is the node of ∞^1 of the rational curves R^{n+1} on P^n_{n+3}. This surface $V_2^{(n)}$ is the complete intersection of the spreads $H_{i_7 \ldots i_{n+3}}$, the locus of S_{n-3}'s on $p_{i_7}, \cdots, p_{i_{n+3}}$ to points x' of the Weddle surface determined in S_3' by the projections of $p_{i_1}, \cdots, p_{i_6}$ from the S_{n-4} on $p_{i_7}, \cdots, p_{i_{n+3}}$. The surface $V_2^{(n)}$ contains all the lines $p_i p_j$, and the curve N^n on P^n_{n+3}. It has $(n-1)$-fold points at P^n_{n+3} such that the tangent cone at p_{i_1} is the cone K^{n-1} from p_{i_1} to the N'^{n-1} on the projections of $p_{i_2}, \cdots, p_{i_{n+3}}$.*

For $n = 3$ this theorem states the more fundamental properties of the Weddle surface; for $n = 4$ it gives additional properties of the surface $V_2^{11} = V_2^{(4)}$.

21. Mapping of W_3' upon the Kummer manifold K_3 in S_7. We derive from the theta relations one such mapping without attempting to draw any geometric conclusions. The mapping is given by the equations:

$$
\begin{aligned}
&\text{(a)}\quad \{1\cdots 7\}[\{1\cdots 8\}]^{1/2}\cdot\vartheta^2\Pi^6 w\cdot\vartheta^2\\
&\qquad=[\pm(17)\{8;\,234\}\cdot\{1;\,56\}H_1\pm\cdots\\
&\qquad\quad\pm(47)\{8;\,123\}\cdot\{4;\,56\}H_4]\,G^2{}_{1\ldots 7};\\
&\text{(b)}\quad \{1\cdots 7\}[\{123568\}(47)]^{1/2}\cdot\vartheta^2\Pi^6 w\cdot\vartheta^2{}_{47}\\
&\qquad=[\pm(16)\{8;\,23\}\cdot(15)G_{147}H_1\pm\cdots\\
&\qquad\quad\pm(36)\{8;\,12\}\cdot(35)G_{347}H_3]G_{12356}G_{1\ldots 7};\\
&\text{(c)}\quad \{1\cdots 7\}[\{5678\}\{1234\}]^{1/2}\cdot\vartheta^2\Pi^6 w\cdot\vartheta^2{}_{5678}\\
&\qquad=[\pm(57)(68)\cdot H_7G_{12347}\pm(56)(78)\cdot H_6G_{12346}]G_{567}G_{1\ldots 7};\\
&\text{(d)}\quad \{1\cdots 7\}[\{2\cdots 7\}(18)]^{1/2}\cdot\vartheta^2\Pi^6 w\cdot\vartheta^2{}_{18}=H_1G^2{}_{1\ldots 7}.
\end{aligned}
\tag{77}
$$

In these equations we have on the left the four types of theta squares, ϑ^2, $\vartheta^2{}_{47}$, $\vartheta^2{}_{5678}$, $\vartheta^2{}_{18}$ each multiplied by the factor $\vartheta^2\Pi^6 w$, and by a modular factor which varies somewhat from type to type. On the right the brackets properly interpreted represent projective loci. Thus in (a) the terms $\{1;\,56\}H_1=(15)(16)H_1,\cdots,\{4;\,56\}H_4$ are all of degrees (4, 4, 4, 4, 3, 3, 3) in $p_1,\cdots,p_7$ respectively when (15) is replaced by (23467), etc., and of degree 4 in x; whereas the coefficients $(17)\{8;234\},\cdots,(47)\{8;123\}$ have ratios which are double ratios of the fundamental octavic.

The equation (d) is a consequence of the definitions (67a, b), (68a). The equation (c) is obtained from the identity B4b′ in (68d). The equation (b) is a transcription of one of the identities A in **17** which reads:

$$
\begin{aligned}
[\{123\}\{856\}]^{1/2}\,\vartheta\vartheta_{47}\pm[\{238\}\{156\}]^{1/2}\vartheta_{2356}\vartheta_{18}\pm[\{138\}\{256\}]^{1/2}\vartheta_{3156}\vartheta_{28}\\
\pm[\{128\}\{356\}]^{1/2}\vartheta_{1256}\vartheta_{38}\equiv 0.
\end{aligned}
$$

The equation (a) is a transcription of a five-term identity:

$$
c^2{}_{1234}\vartheta^2\pm c^2{}_{2348}\vartheta^2{}_{18}\pm c^2{}_{1348}\vartheta^2{}_{28}\pm c^2{}_{1248}\vartheta^2{}_{38}\pm c^2{}_{1238}\vartheta^2{}_{48}\equiv 0.
$$

In this mapping system the factor $G_{1\ldots 7}$ appears throughout on the right. It has been introduced to make the system of order 10 with 6-fold points at $P_7{}^4$. If the involution $I_{1\ldots 6}$ is applied to the system this extraneous factor would be reduced to G_7—the directions at p_7—and would disappear from the projective equations of the mapping system.

§ 5

THE GENERALIZED WEDDLE, W_3, IN S_5.

22. Hyperelliptic theta relations ($p=3$) as projective relations in S_5. We set, as in (47),

$$
\begin{aligned}
&\Pi=\vartheta\vartheta_{18}\cdots\vartheta_{78},\\
&\pi_{8i_1\ldots i_{2k}}=\vartheta_{i_1\ldots i_{2k}}\cdot\Pi^k/\vartheta\vartheta_{8i_1}\cdots\vartheta_{8i_{2k}},\\
&\pi_{i_1\ldots i_{2k+1}}=\vartheta_{8i_1\ldots i_{2k+1}}\cdot\Pi^k/\vartheta_{8i_1}\cdots\vartheta_{8i_{2k+1}}\\
&\qquad(k=0,\cdots,3;\ i_j=1,\cdots,7).
\end{aligned}
\tag{78a}
$$

Then, as observed in (61),

$$\pi i_1 \dots i_{2k+1} \pi_8 i_{2k+2} \dots i_7 = \Pi^2 \cdot \vartheta^2{}_{8 i_1 \dots i_{2k+1}} = \Pi^2 \cdot \vartheta^2{}_{i_{2k+2} \dots i_7}. \tag{78b}$$

We pass from the functions π to the projective forms by setting

$$G_{i_1 \dots i_{2k+1}} = \pi_{i_1 \dots i_{2k+1}} \cdot \{i_1 \cdot \cdot \cdot i_{2k+1}\}^{1/2}; \tag{78c}$$

and pass from binary differences $(i_1 i_2)$ to senary determinants in S_5 according to

$$(i_1 i_2) = \lambda \cdot \epsilon_{i_1 i_2 i_3 \dots i_8} \quad (i_3 \cdot \cdot \cdot i_8). \tag{78d}$$

Since $\vartheta_{i_1 \dots i_{2k}} = \vartheta_{8 i_{2k+1} \dots i_7}$ is the significant factor both in $\pi_{8 i_1 \dots i_{2k}}$ and in $\pi_{i_{2k+1} \dots i}$, a number of alternative projective forms can be obtained from the same theta relation. Furthermore, in the projective situation, p_8 is no longer isolated, so that relations *Bi*, given as distinct in **17**, may yield the same projective relation in S_5. Thus we find from the relations of **17** as indicated that:

$$\begin{aligned}
&\textbf{B3, B1b:} && (14)\,G_{234} \pm (24)\,G_{314} \pm (34)\,G_{124} \equiv 0; \\
&\textbf{B1, B2} : && G_{238} G_{148} \pm G_{318} G_{248} \pm G_{128} G_{348} \equiv 0; \\
&\textbf{B4, B5} : && \pm (14)\,(23)\,G_{248} G_{138} \pm (24)\,(13)\,G_{148} G_{238} \equiv G_{12348}; \\
&\textbf{B4} : && \pm (14)\,(23)\,G_{13567} G_{24567} \pm (24)\,(13)\,G_{14567} G_{23567} \equiv G_{567} G_{1 \dots 7}.
\end{aligned} \tag{78e}$$

In these the binary determinants are to be replaced by senary determinants to secure homogeneity in each point of $P_8{}^5$. The first identity shows that

$$G_{123} = \pm (45678x);$$

the third that G_{12348} is the quadric with nodal plane on p_5, p_6, p_7 and on $p_1, \cdot \cdot \cdot, p_4$, p_8; the fourth that $G_{1 \dots 7}$ is the cubic spread with triple point at p_8 and double curve N^5; and the second is a projective identity. They all can be obtained by the Clebsch transference principle from corresponding identities in S_4.

The degrees in the points of $P_8{}^5$, and in x, of G_{123}, G_{12345}, and $G_{1 \dots 7}$ are respectively

$$(0\,0\,0\,1\,1\,1\,1\,1;1), \quad (2\,2\,2\,2\,2\,4\,4\,4;2), \quad (6\,6\,6\,6\,6\,6\,6\,9;3).$$

If we set

$$H_{78} = \pm \begin{vmatrix} G_{346} G_{126} & G_{345} G_{125} \\ G_{426} G_{136} & G_{425} G_{135} \end{vmatrix} = \vartheta^2{}_{78} \vartheta^2 \cdot \Pi^2 w \cdot [\{123456\}(78)]^{1/2},$$

$$H_{67} = \pm \begin{vmatrix} G_{348} G_{128} & G_{345} G_{125} \\ G_{428} G_{138} & G_{425} G_{135} \end{vmatrix} = \vartheta^2{}_{68} \vartheta^2{}_{78} \cdot \Pi^2 w \cdot [\{123458\}(67)]^{1/2}, \tag{78f}$$

where w is defined in (68g), the projective and the analytic expression for H_{78}, H_{67} are reconciled as before by the identity,

(78g) B2b: $H_{78}G_{347}G_{127} \pm H_{68}G_{346}G_{126} \pm H_{58}G_{345}G_{125} \equiv 0.$

The degrees of H_{78} in P_8^5 and x are (2 2 2 2 2 2 4 4; 4). From (78f) there follows that

(78h) $\pm H_{78}H_{56}/(78)(56) = \pm H_{68}H_{75}/(68)(75) = \pm H_{58}H_{67}/(58)(67).$

In these equations the negative ratios of (78)(56), (68)(75), (58)(67) are to be considered as double ratios of the fundamental octavic. The equations are those of octavic spreads which contain W_3, though we shall find later sextic spreads on W_3.

We need not develop further projective relations as the more important properties of W_3 either have been obtained, or else can be inferred from the properties of its projection W_3'.

23. Geometric properties of W_3. It is comparatively easy to determine the projective equations of the involution I with locus of fixed points, W_3. If $p_1, \cdots, p_8$ are taken as reference points; $G_{178}, \cdots, G_{678}$, as reference S_4's, a provisional form of I is

$$G'_{178} = k_1 G_{23456}/G_{2345678}, \cdots, G'_{678} = k_6 G_{12345}/G_{1234578}.$$

We determine the k_i from the fact that

$$(28)G'_{178} \pm (18)G'_{278} \pm (78)G'_{128} \equiv 0,$$

and $$(28)G_{23456}G_{1345678} \pm (18)G_{13456}G_{2345678} \pm (78)G_{34567}G_{1234568} \equiv 0.$$

A comparison of the first two terms shows that $k_1 = \pm k_2$. Hence

(79) *The equations of the involution $I = I_{1\ldots8}$ are*

$$G'_{178} = \pm G_{23456}/G_{2345678}, \cdots, G'_{678} = \pm G_{12345}G_{1234578}.$$

Thus W_3 is on all the sextic spreads with 4-fold points at P_8^5 of the form

$$G_{178}G_{13456}G_{2345678} \pm G_{278}G_{23456}G_{1345678} = 0.$$

If this spread is recast into theta form, it becomes the identity,

$$\vartheta_{17}\vartheta_{27}\vartheta_{18}\vartheta_{28} - \vartheta_{27}\vartheta_{17}\vartheta_{28}\vartheta_{18} \equiv 0.$$

This double translation of a theta function into G's yields a number of types of sextic spreads on W_3 of which we note the following:

$$
\begin{aligned}
&G_{234}G_{156}G_{13478}G_{25678} \pm G_{134}G_{256}G_{23478}G_{15678} \equiv 0,\\
(80)\qquad &G_{126}G_{678}G_{345}G_{1234578} \pm G_{12678}G_{12345}G_{34578} \equiv 0,\\
&G_{268}G_{167}G_{13458}G_{23457} \pm G_{168}G_{267}G_{23458}G_{13457} \equiv 0.
\end{aligned}
$$

(81) *The variety W_8 has the order* 19. *On it the points of P_8^5 are 9-fold; the lines p_ip_j and N^5 are 3-fold; and the F-loci of the third kind of P_8^5, the* 56 *planes $p_i\ p_j\ p_k$, and the* 8 *cones K_i^4 from respectively p_i to the points of N^5, are simple loci.*

For, W_8 is projected from p_8 into the 10-ic, W_8', with 6-fold points at P_7^4 and triple lines p_ip_j. The directions at p_8, determined by $\vartheta = 0$, become on W_8' the 2-way cut out by the cubic locus of bisecants of the N^4 on P_7^4. This cuts W_8' in a 2-way of order 30 with 12-fold points at P_7^4 from which the seven cubic cones C_i^3 factor, leaving a 9-ic 2-way with triple points at P_7^4. Hence p_8 is a 9-fold point on W_8 and the order of W_8 is 19.

W_8 is cut by the P-locus, G_{123}, in a 2-way of order 19 with 9-fold points at $p_4, \cdots, p_8$ from which ten planes, $p_4p_5p_6$, etc. factor leaving a 9-ic 2-way with triple points at $p_4, \cdots, p_8$, and also with triple points at the three points where G_{123} is cut by the lines p_1p_2, p_1p_3, p_2p_3. The complementary P-locus, G_{45678}, must cut W_8 in this same 9-ic 2-way whose transcendental equation is $\vartheta_{1238} = 0$. This quadric, G_{45678}, cuts W_8 in a 38-ic 2-way from which there must separate the plane $p_1p_2p_3$ twice, the 15 planes like $p_1p_2p_4$ each once, and the three cones like K_1^4 each once, leaving the 9-ic 2-way with triple points at $p_4, \cdots, p_8$.

W_8 is cut by the P-locus, $G_{1\ldots7}$, in a 57-ic 2-way with 18-fold points at $p_1, \cdots, p_7$ and, normally, a 27-fold point at p_8. This 57-ic consists of the 21 planes like $p_8p_1p_2$, and the seven cones like K_1^4, and the cone K_1^8 twice. On this 2-way p_8 has the multiplicity 36 corresponding to the fact that $G_{1\ldots7}$ contains at p_8 all the directions on W_8 at the 9-fold point p_8.

We observe also that

(82) *W_8 contains the surface $V_2^{(5)}$ of* (76), *a 2-way of order* 15 *with 4-fold points at P_8^5.*

We return to the theorem (31a) concerning the linear system Σ of cubic spreads with nodes at P_8^5. The system Σ contains a linear system Σ_8 consisting of all the members of Σ which contain the F-locus of the second kind, $\pi^{(2)}{}_{1\ldots8}$, the locus of bisecants of N^5; or, which contain the paired F-locus of the second kind, $N^5 = \pi^{(2)}$, to a multiplicity two, one greater than the normal multiplicity of Σ. The system Σ_8 contains eight members, $G_{2345678}$,

$\cdots, G_{1234567}$, which lie in a one-parameter cubic system with parameters $t = t_1, \cdots, t_8$. The entire system Σ_3 contains the eight F-loci of the third kind, $K_1^4, \cdots, K_8^4$. A member of Σ such as $G_{123} \cdot G_{45678}$ contains three of these only; namely, K_1^4, K_2^4, K_3^4. Also an F-locus of the third kind of type $\pi^{(3)}_{123}$ is on three of the above eight members of Σ_3; namely, $G_{2345678}, G_{1345678}, G_{1245678}$. In mapping W_3 upon the generalized Kummer 3-way, K_3, in S_7 by means of the system Σ, the $G_{2\ldots8}$ and $G_{123}G_{4\ldots8}$ become singular tangent spaces of K_3; and the F-loci of the third kind become the singular points of K_3. Thus $K_1^4, \cdots, K_8^4$ are mapped upon eight points in an S_3 in S_7 which lie upon a cubic curve, the map of the cone K_t^4. Also $G_{23\ldots8}, \cdots, G_{12\ldots7}$ are mapped upon eight S_6's containing this S_3, and projecting from this S_3 into eight planes of a cubic curve with parameters $t_1, \cdots, t_8$. Hence (cf. [2] p. 310).

(83) *The Kummer 3-way in S_7 with 64 singular points and 64 singular tangent spaces has, in the hyperelliptic case, an additional configuration of 64 S_3's each with the following properties. An S_3 is on 8 singular points which lie on a cubic space curve with parameters $t = t_1, \cdots, t_8$; and is on 8 singular spaces which envelop a cubic space curve with parameters $t = t_1, \cdots, t_8$. Each of the remaining 56 singular points (tangent spaces) is on three of the 8 singular tangent spaces (points).*

The existence of this self-dual configuration connected with K_3 is a consequence of the existence of the F-loci of the second kind, intermediate between those of the first and third kinds, in the space of the Weddle W_3.

§ 6

THE HYPERELLIPTIC ($p = 3$) FORM OF THE CAYLEY DIANODE SURFACE.

24. Definition and equation of the surface. A singular space of the Kummer 3-way, K_3^{24}, in S_7 touches K_3 along a 2-way, M_2^{12}. If the functions are not hyperelliptic, the M_2^{12} is the map of Cayley's dianode surface, a sextic surface T with triple points at seven points, P_7^3, in ordinary space, by its linear system of adjoint quartic surfaces [cf. [1] p. 187 (1)]. Schottky has given a parametric equation of the dianodal surface in terms of theta functions of order four for those values u which satisfy $\vartheta(u) = 0$ (cf. [1] p. 181). However, the situation is quite different in the hyperelliptic case. Since we hope to make in the future a closer study of the hyperelliptic Kummer manifold K_p with the aid of the mappings indicated in this article, it seems pertinent to develop here the peculiarities of Schottky's representation in the hyperelliptic case.

The representation is based on the following theta relations of types A and C [cf. (17)]:

$$\vartheta(u) = 0;$$

$$[\{238\}\{156\}]^{1/2}\,\vartheta_{1478}\vartheta_{18} \pm [\{318\}\{256\}]^{1/2}\,\vartheta_{2478}\vartheta_{28} \pm [\{128\}\{356\}]^{1/2}\vartheta_{3478}\vartheta_{38} = 0;$$

(84a)
$$[(12)(34)(58)(67)]^{1/2}\vartheta_{23}\vartheta_{14} \pm [(14)(23)(57)(68)]^{1/2}\vartheta_{1278}\vartheta_{3478} \pm [(14)(23)(56)(78)]^{1/2}\,\vartheta_{1268}\vartheta_{3468} = 0;$$

$$[\{234\}\{856\}]^{1/2}\,\vartheta_{1478}\vartheta_{48} \pm [\{438\}\{256\}]^{1/2}\,\vartheta_{1247}\vartheta_{24} \pm [\{428\}\{356\}]^{1/2}\,\vartheta_{1347}\vartheta_{34} = 0.$$

The functions introduced by Schottky are

(84b)
$$\begin{aligned} \Pi &= \vartheta_{18}\vartheta_{28}\cdot\cdots\vartheta_{78};\\ P_{ijk} &= \vartheta_{ijks}\vartheta_{is}\vartheta_{js}\vartheta_{ks};\\ P_{i,j} &= \vartheta_{ij}\cdot\Pi\vartheta_{is}/\vartheta_{js};\\ P_{ijkl} &= \vartheta_{mnos}\cdot\Pi^2/\vartheta_{ms}\vartheta_{ns}\vartheta_{os};\\ P_i &= \vartheta^2{}_{is}\cdot\Pi^2\\ &(i,j,\cdots,o = 1,\cdots,7). \end{aligned}$$

We convert some of these into projective forms by setting

(84c)
$$\begin{aligned} G_{123} &= \pm P_{123}\cdot\{123\}/[\{8;123\}\{4567\}]^{1/2},\\ G_{1,2} &= \pm P_{1,2}\cdot\{134567\}\{34567\}/[\{234567\}\{8;134567\}(23)]^{1/2}. \end{aligned}$$

The first relation (84a) then yields

$$\{2347\}G_{147} \pm \{3147\}G_{247} \pm \{1247\}G_{347} \equiv 0.$$

This shows, in the customary way, that the functions G_{123} are linear sections of a theta manifold in S_3 by planes on the points p_1, p_2, p_3 respectively of a set $P_7{}^3$. Moreover the coefficients of the relations are such that the $P_7{}^3$ is on a norm-curve N^3 with parameters $t_1,\cdots,t_7$. If N^3 is taken as

(84d)
$$\begin{aligned} &x_0 = t^3,\quad x_1 = t^2,\quad x_2 = t,\quad x_3 = 1, \qquad \text{then}\\ &G_{147} = \pm(147x),\quad \{2347\} = \pm(2347). \end{aligned}$$

With this transition to projective quaternary determinants the second relation (84a) becomes

(84e)
$$\begin{aligned} G_{3,2} &= G_{356}G_{347}(1357)(1346) - G_{357}G_{346}(1356)(1347)\\ &= (13)^2\{13;4567\}[G_{356}G_{347}(57)(46) - G_{357}G_{346}(56)(47)]. \end{aligned}$$

The first expression for $G_{3,2}$ is purely projective, and shows that $G_{3,2}$ is a quadric with node at p_3 and simple points at p_4, p_5, p_6, p_7. From the symmetry of its definition in (84c), the quadric $G_{3,2}$ must pass through p_1 also.

But since in the present case P_7^3 is on N^3, $G_{3,2}$ must contain N^3, and therefore also the point p_2. Thus it must differ from $G_{3,1}$ by a constant factor only, and we find at once from the second form of $G_{3,2}$ in (84e) that

$$(84f) \qquad (2357)(2346)G_{3,2} \pm (1357)(1346)G_{3,1} \equiv 0.$$

If this projective identity is recast into transcendental form it becomes

$$(84g) \qquad [(82)(31)]^{1/2}\vartheta_{23}\vartheta_{18} \pm [(81)(23)]^{1/2}\vartheta_{31}\vartheta_{28} \equiv 0,$$

which is the form the identity B4b in **17** takes when $\vartheta(u) = 0$.

In order to give $G_{3,2}$ in (84e) a specific sign we shall write

$$(84h) \qquad \begin{aligned} G_{147} &= (147x) \\ G_{3,2} &= W(3;\ 14567x), \end{aligned}$$

and agree that corresponding functions shall arise from the latter by even permutations of the arguments $1, \cdots, 7, x$. Thus $W(x;\ 123456) = \pm G_{x,7}$ is the Weddle quartic surface with nodes at $p_1, \cdots, p_6$, and is to be taken with the specific sign which arises when the permutation, $(27)(3x)$, is applied to $G_{3,2}$ in (84e). The function W is an alternating function of its last six arguments.

The third of the relations (84a) can be recast into

$$(84i) \qquad \begin{aligned} P_{2356} \cdot \{8;17\}\{2317\}\{2356\}\{235617\}/[\{12345678\}\{147\}]^{1/2} \\ \equiv \pm (8137)G_{356}G_{2,4} \pm (8127)G_{256}G_{3,4}. \end{aligned}$$

The right member is purely projective. Hence P_{2356} is a cubic surface with nodes at p_2, p_3, p_5, p_6 and on p_1, p_7; but also on p_4 since $G_{2,4}$, $G_{3,4}$ are also on p_4. When P_7^3 is on N^3 there is a pencil of such surfaces (the image of a pencil of planes under a regular cubic transformation). If p_8 is taken on N^3 with parameter $t = t_8$, the binary factors (81) and (82) separate from (8137) and (8127) leaving t_8 as the linear parameter of the pencil.

The identical relations ([1] p. 183) among the functions now lead in part to projective identities. For example, the first one, $P_{1,2}P_{2,3}P_{3,1} = P_{2,1}P_{1,3}P_{3,2}$, reduces by the use of (84b, c) to $G_{1,2}G_{2,3}G_{3,1} = \pm G_{2,1}G_{1,3}G_{3,2}$, which, according to (82f), is an identity. But the second relation,

$$(85a) \qquad P_{3,4}P_{124}P_{4567} - P_{4,3}P_{123}P_{3567} \equiv 0,$$

is converted by the use of (84c, e, i) into the projective relation,

$$(85b) \qquad \begin{aligned} S \equiv\ & G_{3,4}G_{124}\,[(1278)G_{457}G_{6,3} \mp (1268)G_{456}G_{7,3}] \\ & \mp G_{4,3}G_{123}\,[(1278)G_{357}G_{6,4} \mp (1268)G_{356}G_{7,4}] \equiv 0. \end{aligned}$$

This is a relation of the sixth order, an identity in u for values subject to $\vartheta(u) = 0$, but not an identity in x for all positions of p_8 on N^3; and therefore

is the equation of the sextic surface whose parametric equations are furnished by (84c). For variable $p_8 = t_8$ on N^3, the factors (81), (82) separate from (1278) and (1268), so that the sextic surface is a member of a pencil with linear parameter t_8. The degrees of each term of the equation in $p_1, \cdots, p_8, x$ are 6, 6, 4, 4, 5, 6, 6, 1, 6.

On isolating the parameter, (1278) : (1268), of the pencil it reads:

$$(85c) \qquad \begin{aligned} S = &(1278)[G_{3,4}G_{6,3}G_{124}G_{457} \mp G_{4,3}G_{6,4}G_{123}G_{357}] \\ &\mp (1268)[G_{3,4}G_{7,3}G_{124}G_{456} \mp G_{4,3}G_{7,4}G_{123}G_{356}] = 0. \end{aligned}$$

Each term is quadratic in $G_{i,j}$ and therefore the pencil contains N^3 doubly. Each term contains $G_{4,3}$ or $G_{3,4}$ and therefore the pencil contains the line p_3p_4. It must therefore contain each of the 21 lines p_ip_j. Each of the two terms of the first member of the pencil contains explicitly all of the 21 lines which are on p_6, p_3, p_4 as well as the lines p_1p_2, p_5p_7, leaving for examination only the lines from p_1, p_2 to p_5, p_7. With the definition of sign in (84e, h) we find, when x is taken to be $\lambda_1 p_1 + \lambda_2 p_5$, that the term in $\lambda_1{}^3\lambda_2{}^3$ cancels in (85c) only when the ambiguous sign within the brackets is negative, or when the ambiguous sign outside the brackets in (85b) is negative.

The same determination of sign permits of writing the more specific form of (84f):

$$(85d) \qquad (2357)(2346)G_{3,2} + (1357)(1346)G_{3,1} \equiv 0.$$

We use this relation to remove the factor $G_{6,4} = kG_{6,3}$ from the first member of the pencil (85c). It becomes, to within the factor $-(1278)G_{6,4}/(6312)(6357)$,

$$\{(6412)(6457)G_{3,4}G_{124}G_{457} + (6312)(6357)G_{4,3}G_{123}G_{357}\}.$$

We prove that this brace is a Weddle surface as follows: replace $G_{3,4}$ and $G_{4,3}$ in the brace by the values,

$$\begin{aligned} G_{3,4} &= (6357)(6312)G_{351}G_{327} - (6351)(6327)G_{357}G_{321}, \\ G_{4,3} &= (6457)(6412)G_{452}G_{417} - (6452)(6417)G_{457}G_{412}. \end{aligned}$$

If now the quaternary determinants are all expressed in terms of the binary differences the factor

$$m = \{6;\ 123457\}^2\{34;\ 1257\}(12)(57)$$

can be removed. The two like products of four G's then have further coefficients (51)(27), (25)(17) which coalesce into (57)(21). The brace then becomes

$$m \cdot (12)(57) \cdot [-G_{124}G_{574}G_{357}G_{312} - G_{124}G_{574}G_{351}G_{327} + G_{123}G_{573}G_{174}G_{524}].$$

The first two terms in the bracket coalesce into $G_{124}G_{574}G_{352}G_{371}$, and the bracket is therefore the Weddle quartic surface $G_{x,6}$ with nodes at the points of P_7^3 other than p_6. Collecting all the terms, and passing to the second member of the pencil (85c) by interchanging 6,7 we find that

$$(85e)\quad S=(4567)[(1278)(1246)G_{6,4}G_{x,6}\mp(1268)(1247)G_{7,4}G_{x,7}].$$

The facts concerning the surface S, and the pencil in which it lies, may now be summarized as follows:

(86) *The pencil of sextic surfaces S has triple points at P_7^3, has the double curve N^3 asymptotic on one sheet of the surface, and contains the* 21 *lines p_ip_j. As p_8 with parameter $t=t_8$ runs along N^3, the surface S for $t=t_i$ $(i=1,\cdots,7)$ degenerates into $G_{i,j}G_{x,i}$. The equation of the surface is obtained from any determinant of this (or similar) matrix,*

$$\begin{vmatrix} (45)(67)G_{5,4}G_{x,5} & (46)(75)G_{6,4}G_{x,6} & (47)(56)G_{7,4}G_{x,7} \\ (85)(67) & (86)(75) & (87)(56) \end{vmatrix}=0,$$

in which the binary differences are to be interpreted as terms of double ratios. With the determinations of sign in (84e, h) *the upper signs are to be chosen in* (85b, c, e).

For, we see in (85e) that when $p_8=p_6$, S becomes $G_{6,4}G_{x,6}$. The cone $G_{6,4}$ contains N^3, and the Weddle surface $G_{x,6}$ contains N^3 as an asymptotic curve. We shall verify in the next section that the sum of the elements in the upper row of the given matrix is identically zero [cf. (87e)], and evidently this is true of the lower row also. The last determinant of the matrix reduces to (85e) with upper sign. Since S also admits similar representations in terms of the other determinants, the three representations can coexist only for the given choice of signs.

We observe that the base curve of order 36 of the pencil consists of N^3 five-fold, and the 21 lines p_ip_j.

25. The surface S in a binary notation with reference to N^3. We use that coördinate system in S_3 in which the coördinates of a point are the coefficients of a binary cubic and perfect cubes are points on N^3. The notation is that of Gordan ([13] Part II, pp. 172-3) in which

$$(87a)\quad \begin{aligned} &f=(a_1t)^3=(a_2t)^3=\cdots;\\ &\Delta=(a_1a_2)^2(a_1t)(a_2t)=(\Delta t)^2=(\Delta_1t)^2=\cdots;\\ &Q=(a_1a_2)^2(a_3a_2)(a_1t)(a_3t)^2=(Qt)^3=(Q_1t)^3=\cdots;\\ &R=(\Delta\Delta_1)^2;\\ &2Q^2+Rf^2+\Delta^3\equiv 0;\ \text{and let also}\\ &k=f\cdot Q=(kt)^6=(k_1t)^6=\cdots. \end{aligned}$$

The quaternary determinants then reduce to binary products as in

(87b) $$G_{123} = \{123\} \cdot (a_1 t_1)(a_1 t_2)(a_1 t_3)\,; \quad (1234) = \{1234\}.$$

The cone $G_{3,2}$ in (84e), and the Weddle surface $G_{x,3}$, are now

(87c) $$\begin{aligned} G_{3,2} &= -\{3\,;\,14567\}^2 \{14567\} \cdot (\Delta t_3)^2/2, \\ G_{x,3} &= c \cdot \{124567\} \cdot (k\alpha_3)^6, \end{aligned}$$

where the numerical constant c need not be evaluated, and where $(\alpha_3 t)^6$ is the sextic with roots $t_1, \cdots, t_7$ excepting t_3.

We first prove that

(87d) $$\begin{aligned} (67) \cdot (\Delta t_5)^2 \cdot (kt_6)(kt_7)(kt)^4 + (75) \cdot (\Delta t_6)^2 \cdot (kt_7)(kt_5)(kt)^4 \\ + (56) \cdot (\Delta t_7)^2 \cdot (kt_5)(kt_6)(kt)^4 \equiv 0. \end{aligned}$$

For, the left member changes sign when any two of t_5, t_6, t_7 are interchanged, and therefore contains the factor $(56)(57)(67)$. The residual factor is a covariant of the cubic, of order 4 and degree 6, which cannot exist.

If the identity be polarized with respect to t_1, t_2, t_3, t_4, it becomes by virtue of (87c),

(87e) $$(45)(67) G_{5,4} G_{x,5} + (46)(75) G_{6,4} G_{x,6} + (47)(56) G_{7,4} G_{x,7} \equiv 0.$$

We now prove the theorem:

(88a) *The pencil of sextic surfaces* (86) *may be defined as that pencil with triple points at P_7^3 only on N^3, whose members are individually invariant under the cubic Cremona involution J whose generic pair x, x' is on a bisecant of N^3 and harmonic with the intersections of the bisecant.*

Under J the points represented by f and Q are interchanged and then ([13] § 149)

$$f' = Q, \quad \Delta' = R\Delta/2, \quad Q' = -Rf^2/4, \quad R' = R^3/4.$$

The P-surface of J is the quartic surface, $R = 0$, the locus of tangents of N^3, on which N^3 is a double doubly-asymptotic curve (cf. [14] p. 4). If J is to leave a sextic surface unaltered, it must transform the surface into itself multiplied by R^3. The only independent covariants of f of degree 6 which acquire a factor R^3 are

$$f\Delta Q, \quad Rf^2, \quad R\Delta, \quad Q^2$$

which are transformed to within the factor $R^3/8$ into respectively

$$-f\Delta Q, \quad 2Q^2, \quad R\Delta, \quad Rf^2/2.$$

Hence

$$R\Delta, \quad Rf^2 + 2Q^2, \quad Rf^2 - 2Q^2, \quad f\Delta Q$$

are transformed respectively into

$$R\Delta, \quad Rf^2 + 2Q^2, \quad -(Rf^2 - 2Q^2), \quad -f\Delta Q.$$

Hence, replacing Rf^2 by using the syzygy (87a),

(88b) *Under J there are two systems of invariant sextic surfaces, namely,*

$$(R\Delta, \alpha)^2 + (\Delta^3, \beta)^6 = 0,$$
$$(f\Delta Q, \gamma)^8 + (4Q^2 + \Delta^3, \delta)^6 = 0,$$

where the coefficients of $(\alpha t)^2$, $(\beta t)^6$, $(\gamma t)^8$, $(\delta t)^6$ in the indicated apolarity invariants are the parameters of the systems.

The coefficients of Δ contain N^3 simply, those of Q contain N^3 as a simple simply-asymptotic curve. Hence the first system contains N^3 as a triple curve. The second system, however, contains N^3 as a double simply-asymptotic curve. We wish to find the conditions on $(\gamma t)^8$, $(\delta t)^6$ that a surface of this second system may have a triple point at $t = t_1$. The part containing Δ^3, which has the triple curve N^3, may be dropped, and we have to examine

$$(a_1\gamma)^3(a_2a_3)^2(a_2\gamma)(a_3\gamma)(a_4a_5)^2(a_6a_5)(a_4\gamma)(a_6\gamma)^2$$
$$+ 4(a_1a_2)^2(a_3a_2)(a_1\delta)(a_3\delta)^2(a_4a_5)^2(a_6a_5)(a_4\delta)(a_6\delta)^2 = 0.$$

The quadric polar of the point (t_1t) is obtained by setting four of the six symbols, a_{i0}, a_{i1} (in each of 15 ways) equal to t_{11}, $-t_{10}$, leaving the other two as variable, and dividing by 15. The result is

$$2[(\gamma t_1)^7(a_1\gamma)(a_1t_1)^2 + 2(a_1t_1)^3 \cdot (\delta t_1)^6] \cdot (a_2t_1)^3/15.$$

The factor, $(a_2t_1)^3$, the osculating plane of N^3 at t_1, accounts for the fact that N^3 is a simply-asymptotic curve. The vanishing of the bracket is the significant part of the condition on $(\gamma t)^8$, $(\delta t)^6$ that the surface may have a triple point at t_1. On replacing symbols a_1 by t and factoring out $(t_1t)^2$, this condition on γ, δ reduces to the identity, linear in t,

$$(\gamma t_1)^7(\gamma t) + 2(\delta t_1)^6 \cdot (t_1t) \equiv 0.$$

On setting $t = t_1$ in this identity, $(\gamma t_1)^8 = 0$.

Thus to secure triple points at P_7^3 we set

(88c)
$$(\sigma t)^7 = (t_1t) \cdot \cdots \cdot (t_7t),$$
$$(\gamma t)^8 = (t_8t) \cdot (\sigma t)^7,$$

the t_8 remaining in γ as a linear parameter. Then

$$8(\gamma t_1)^7(\gamma t) = \{2345678\,;\,1\} \cdot (t_1t).$$

In order to satisfy the above linear identity at P_7^3, the coefficients of $(\delta t)^6$ must satisfy the seven linear non-homogeneous equations,

$$16(\delta t_1)^6 = \{1;\ 2345678\}, \cdots, 16(\delta t_7)^6 = \{7;\ 1234568\}.$$

Now $7(\sigma t)^6(\sigma t_8) = \Sigma(18)(t_2 t)\cdots(t_7 t)$, and $7(\sigma t_1)^6(\sigma t_8) = \{1;\ 2345678\} = 16(\delta t_1)^6$, etc. Hence $(\delta t)^6 = 7(\sigma t_8)(\sigma t)^6/16$.

(88d) *With* $(\gamma t)^8$ *and* $(\sigma t)^7$ *defined as in* (88c) *the pencil (for variable* t_8*) of sextic surfaces,*

$$16\,[f\Delta Q, \gamma]^8 + 7\,[4Q^2 + \Delta^3, (\sigma t_8)(\sigma t)^6] = 0,$$

has triple points at $P_7{}^3$ *only on* N^3*; and each member is invariant under* J.

In order to complete the proof of (88a) it is necessary to show that the member of the pencil (88d) obtained by setting $t_8 = t_1$ is the product of the quadric cone on N^3 with vertex at p_1, and the Weddle with nodes at $p_2, \cdots, p_7$, i. e., is

$$(\Delta t_1)^2 \cdot (fQ, \alpha_1)^6 = (\Delta t_1)^2 \cdot (k\alpha_1)^6 \qquad \text{[cf. (87c)]}.$$

When $t_8 = t_1$, $(\gamma t)^8 = (t_1 t)^2 \cdot (\alpha_1 t)^6$, $(\sigma t_1)(\sigma t)^6 = (6(t_1 t) \cdot (\alpha_1 t_1)(\alpha_1 t)^5/7$. The member of the pencil (88d) is then

$$8[\Delta k,\ (t_1 t)^2 \cdot (\alpha_1 t)^6]^8 + 3[4Q^2 + \Delta^3,\ (t_1 t) \cdot (\alpha_1 t_1)(\alpha_1 t)^5]^6$$
$$= 8[\{\Delta k\}_{t_1{}^2},\ (\alpha_1 t)^6]^6 + 3[\{4Q^2 + \Delta^3\}_{t_1},\ (\alpha_1 t_1)(\alpha_1 t)^5]^5 = 0,$$

where subscripts $t_1{}^2$, t_1 indicate polarizations. The second polar form can be expanded as follows:

$$\{\Delta k\}_{t_1{}^2} = (\Delta t_1)^2 \cdot (kt)^6 - 3(\Delta k)(\Delta t)(kt_1)(kt)^4 \cdot (t_1 t)/2$$
$$+ 15(\Delta k)^2 (kt)^4 \cdot (t_1 t)^2/28.$$

The first term of this expansion contributes the desired product,

$$8(\Delta t_1)^2 \cdot (k\alpha_1)^6.$$

The last term, involving a covariant, $(\Delta k)^2(kt)^4$, of order 4 and degree 6, vanishes identically. Since the last term does vanish identically, the middle term can be written as $-3\{(\Delta k)(\Delta t)(kt)^5\}_{t_1} \cdot (t_1 t)/2$. But the transvectant

$$(\Delta k)(\Delta t)(kt)^5 = (\Delta, fQ)^1$$
$$= [f(\Delta, Q)^1 + Q(\Delta, f)^1]/2 = -(4Q^2 + \Delta^3)/4$$

(cf. [18]). Hence the member can be written as

$$8(\Delta t_1)^2 \cdot (k\alpha_1)^6 + 3[\{4Q^2 + \Delta^3\}_{t_1} \cdot (t_1 t),\ (\alpha_1 t)^6]^6$$
$$+ 3[\{4Q^2 + \Delta^3\}_{t_1},\ (\alpha_1 t_1)(\alpha_1 t)^5]^5 = 0.$$

In this the last two transvectants cancel and the proof is complete.

We have not yet determined the geometric character of the point $t_8 = p_8$ on the member of the pencil which it determines. This is as follows:

(89) *Each member of the pencil of sextic surfaces S which is not degenerate has a pinch point at one point $p_8 = t_8$ on its double curve.*

For, if, in the quadric polar of $(t_1 t)^3$ above, we replace t_1 by t_8, and omit the osculating plane of N^3, $2(a_2 t_8)^3/15$, the remaining factor,

$$(\gamma t_8)^7 (a_1\gamma)(a_1 t_8)^2 + 2(a_1 t_8)^3 \cdot (\delta t_8)^6,$$

becomes, due to the above determinations of γ and δ, precisely

$$\{1234567; 8\} \cdot (a_1 t_8)^3.$$

This also is the osculating plane of N^3 except at the triple points

$$t_8 = t_1, \cdots, t_8 = t_7.$$

Furthermore this part of the quadric polar of the generic point t of N^3, $(\gamma t)^7 (a_1\gamma)(a_1 t)^2 + 2(a_1 t)^3 \cdot (\delta t)^6 = 0$, reduces to $(a_1 t_8)(a_1 t)^2 \cdot (\sigma t)^7 = 0$ which is the plane on p_8 and tangent to N^3 at t, i. e., the tangent plane at t of the quadric cone, G_8, on N^3 with vertex at p_8. Hence

(90 *The surface S determined by $t_8 = p_8$ has a sheet passing through N^3 and tangent along N^3 to the quadric cone G_8 with vertex at p_8.*

26. Mapping of S upon the section of K_3 by a singular space. From (84b) there follows that

$$\begin{aligned} \Pi^2 \cdot \vartheta^2_{ij} &= P_{i,j} P_{j,i}, \\ \Pi^2 \cdot \vartheta^2_{ijk8} &= P_{ijk} P_{lmno}, \\ \Pi^2 \cdot \vartheta^2_{i8} &= P_i \\ &\qquad (i, j, \cdots = 1, \cdots, 7). \end{aligned} \tag{91}$$

The products $P_{i,j}P_{ji}$ and $P_{ijk}P_{lmno}$ have been converted into quartic surfaces on N^3 with nodes at $P_7{}^3$. We identify P_i with a similar quartic surface in precisely the same way as in ([1] p. 183). With $i = 7$, P_7 is proportional to

$$\{238\}(16)(17)G_{1,6}G_{7,1} \pm \{318\}(26)(27)G_{2,6}G_{7,2} \pm \{128\}(36)(37)G_{3,6}G_{7,3}.$$

If from this we eliminate $G_{7,2}$ and $G_{7,3}$ by using [cf. 84(f)]

$$\begin{aligned} (1734)(1756)G_{7,1} &= \pm(2734)(2756)G_{7,2}, \\ (1724)(1756)G_{7,1} &= \pm(3724)(3756)G_{7,3}, \end{aligned}$$

there results

$$G_{7,1} \cdot [\{234578\}(23)G_{1,6} \pm \{314578\}(31)G_{2,6} \pm \{124578\}(12)G_{3,6}] = 0.$$

On inserting the binary notation from (87c), this becomes

$$(\Delta t_7)^2 \cdot [\{238\} \cdot (\Delta t_1)^2 \pm \{318\} \cdot (\Delta t_2)^2 \pm \{128\} \cdot (\Delta t_3)^2] = 0.$$

Since from its definition P_7 must be symmetrical with respect to $p_1, \cdots, p_6$,

the $\pm$ signs must be such that the second factor of P_7 must be $\pm\{123\}\cdot(\Delta t_8)^2$ whence

$$P_7 = k\cdot(\Delta t_7)^2\cdot(\Delta t_8)^2. \tag{92}$$

Thus P_7, the product of the two quadric cones, G_7, G_8 on N^3 with vertices at p_7 and p_8 respectively, is in the linear system of quartic surfaces which contain N^3 and have nodes at $P_7{}^3$.

Quartic surfaces of this system are found in the system which contains N^3, which is made up as follows:

$$(\Delta f^2,\gamma)^8+(fQ,\delta)^6+(\Delta^2,\epsilon)^4+\zeta R=0. \tag{93}$$

The last two terms comprise those surfaces which contain N^3 doubly so that we seek to determine the forms $(\gamma t)^8$, $(\delta t)^6$ so that the system may have double points at $t_1,\cdots,t_7$ on N^3. The polar plane of the point represented by $(t_1 t)^3$ is $(at_1)^3\cdot(\delta t_1)^6+2(at_1)^2(a\gamma)(\gamma t_1)^7=0$. If this is to vanish for all points, $(at)^3$, it must vanish identically when symbols a are replaced by variables t from $(at')=(tt')$; i. e.,

$$(tt_1)\cdot(\delta t_1)^6-2(\gamma t)(\gamma t_1)^7\equiv 0.$$

For the particular value $t=t_1$, $(\gamma t_1)^8=0$, whence we introduce γ and σ as in (88c), and find as before that $4(\delta t)^6=7(\sigma t_8)(\sigma t)^6$. With these values of γ and δ, the polar plane of a general point t on N^3 has the equation, $(at)^2(at_8)\cdot(\sigma t)^7=0$. If t is not at a point of $P_7{}^3$, this is the plane on t_8 containing the tangent at t, i. e., is the tangent plane of G_8 at t; whereas if t is at t_8, it is the osculating plane of N^3 at t_8. Hence

(94) *The linear system of theta squares, other than $\vartheta^2(u)=0$, are represented on S by the linear system of ∞^6 quartic surfaces,*

$$4[\Delta f^2,(t_8 t)\cdot(\sigma t)^7]^8+7[fQ,(\sigma t_8)(\sigma t)^6]^6+(\Delta^2,\epsilon)^4+\zeta R=0,$$

where $(\sigma t)^7=(t_1 t)\cdots(t_7 t)$, and where ζ and the coefficients of $(\epsilon t)^4$ are the six non-homogeneous parameters of the system. This system contains N^3, touches G_8 along N^3, and at $t_8=p_8$ has for tangent plane the osculating plane of N^3. The system maps S upon the section of the Kummer 3-way, K_3, in S_7 by its singular tangent space, $\vartheta(u)=0$.

A projection of this section of K_3 from three of the singular points of K_3 upon the section, which is a sextic surface in ordinary space, is the subject of an elegant memoir by Humbert.[15] This projection is thus a birational transform of the Cayley dianodal surface. It has of course a much lower degree of symmetry with respect to the theta functions than the Cayley surface. A further development of the properties of the surface of Humbert, in both the general and the hyperelliptic case, is given by L. Remy.[16, 17]

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REFERENCES.

[1] A. B. Coble, "Algebraic Geometry and Theta Functions," *Colloquium Publications of the American Mathematical Society*, New York, Vol. 10 (1929).

[2] A. B. Coble, "The Equation of the Eighth Degree," *Bulletin of the American Mathematical Society*, Vol. 30 (1924), pp. 301-313.

[3] Report, "Selected Topics in Algebraic Geometry," *Bulletin, National Research Council*, No. 63 (April, 1928). Washington.

[4] A. B. Coble, "Point Sets and Allied Cremona Groups," *Transactions of the American Mathematical Society*, I: Vol. 16 (1915), pp. 155-198; II: Vol. 17 (1916), pp. 345-385; III: Vol. 18 (1917), pp. 331-372.

[5] F. Schottky, "Ueber die Beziehungen zwischen den sechzehn Theta-functionen von zwei Variabeln," *Journal für Mathematik*, Vol. 105 (1889), pp. 233-249.

[6] A. B. Coble, "The Ten Nodes of the Rational Sextic and of the Cayley Symmetroid," *American Journal of Mathematics*, Vol 41 (1919), pp. 243-265.

[7] A. B. Coble, "Associated Sets of Points," *Transactions of the American Mathematical Society*, Vol. 24 (1922), pp. 1-20.

[8] A. Krazer, *Lehrbuch der Thetafunktionen*, Teubner, Leipzig (1903).

[9] J. Thomae, "Beitrag zur Bestimmung von $\theta(0, 0, \ldots, 0)$ durch die Klassenmoduln algebraischer Functionen," *Journal für Mathematik*, Vol. 71 (1870), pp. 201-222.

[10] O. Zariski, "On Hyperelliptic Theta-functions with Rational Characteristics," *American Journal of Mathematics*, Vol. 50 (1928), pp. 315-344.

[11] H. F. Baker, "Elementary Note on the Weddle Quartic Surface," *Proceedings of the London Mathematical Society* (2), Vol. 1 (1904), pp. 247-261.

[12] R. W. H. T. Hudson, *Kummer's Quartic Surface*, Cambridge (1905).

[13] P. Gordan, *Invarianten-theorie*, Teubner, Leipzig (1885).

[14] A. B. Coble, "On the Relation between the Three-parameter Groups of a Cubic Space Curve and a Quadric Surface," *Transactions of the American Mathematical Society*, Vol. 6 (1906), pp. 1-20.

[15] G. Humbert, "Sur une surface du sixième ordre liée aux fonctions abéliennes de genre trois," *Journal de Mathématiques*, Vol. 61 (1896), pp. 263-293.

[16] L. Remy, "Sur certaines surfaces algébriques liées aux fonctions abéliennes de genre trois," *Journal de Mathématiques*, Vol. 73 (1908), pp. 1-37.

[17] L. Remy, "Sur une classe de surfaces algébriques liées aux fonctions abéliennes de genre trois," *Annales de l'École Normale Supérieure*, Vol. 45 (1909), pp. 193-274.

On the Moduli of Algebraic Functions.

By Francesco Severi.

(Extract from a letter to Professor Oscar Zariski.)

In your paper, published in this Journal, January 1930, pp. 150-170, you observe that the reasoning employed in sec. 5 of my article "Sul Teorema di Esistenza di Riemann," *Rendiconti del Circolo Matematico di Palermo*, (1922), proves that on a curve Γ of genus $p(>1)$, of general moduli, the two-fold of a *generic* g_ν^1, of any order ν (necessarily $\geqq p/2+1$), is non-special; but that the question remains whether this holds for *any* g_ν^1 of Γ.

I wish to show you that the above theorem is also valid if referred to *any* g_ν^1 of Γ, and that the essential elements necessary to reach this conclusion are already contained in my quoted article and in my "Vorlesungen über algebraische Geometrie," to which that article refers. Your remark is at any rate opportune, since perhaps the deduction was too concise, and therefore it will be useful to develop it with more details, as I now intend to do. In the sequel I use the notations of my article, and I explicitly remind you that the curves, which are considered below, are always supposed to be of genus $p>1$.

First of all a remark of a general character. Let H be an arbitrary irreducible system, ∞^r, of curves C of genus p, and let H contain $\infty^{r'}(r'<r)$ curves C_0, constituting a system H' and enjoying a given property, which is not enjoyed by a generic C (a priori this property need not be invariant under birational transformations). Let the curves C birationally equivalent to a generic C be ∞^δ, and let the curves C_0 of H', which are birationally equivalent to a generic C_0, be $\infty^{\delta'}$. If $\delta' \geqq \delta$ (this is the essential hypothesis!), then denoting by μ, μ' the number of moduli of the variable curve C of H and of the variable curve C_0 of H' respectively, we have evidently the following relations:

$$\mu = r-\delta, \qquad \mu' = r'-\delta' < \mu-(\delta'-\delta),$$

and hence the C_0's, as curves of H, are of special moduli.

In our case, H is the irreducible system of the curves C, of order $n+\nu$ and genus p, having two multiple points of orders ν, n at P, Q respectively *

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* The integer n must satisfy the only condition that series g_n^1 exist on every curve of genus p. Accordingly we must have $n \geqq p/2+1$.

and $d=(n-1)(\nu-1)-p$ distinct ordinary double points; H' is the system of those curves C_0 of H, which enjoy the particular property, that the index of speciality i' of the two-fold of the g_ν^1, cut out on a generic C_0 by the lines of the pencil Q, is *greater* than the index of speciality $i(\geqq 0)$ of the two-fold of the similar g_ν^1, relative to a generic C; $r=2n+2\nu+p-1$; $\delta=2n+2\nu-2p+2+i$. As far as δ' is concerned, I shall show later that

$$\delta' \geqq 2n+2\nu-2p+2+i', \tag{1}$$

whence it follows that

$$\mu' < \mu-(\delta'-\delta) \leqq 3p-3-i'.$$

Hence, in the case of curves of general moduli ($i=0$), the following theorem holds:

On a curve of general moduli of genus p, the two-fold of any *infinite linear series is non-special.*

For $i>0$, the same reasoning, which leads to (1), with a few additional considerations, serves to prove that:

On the most general curve of genus p, containing a g_ν^1, $\nu < p/2+1$, without fixed points, the two-fold of any *series g_ν^1, without fixed points, has its index of speciality $i>0$, $3p-3-i$ being the number of moduli on which that curve depends.*

Moreover it is known, as was shown by my assistant, Professor B. Segre,* and as you yourself found out again in your paper, that the number i for the above curve is $i=p-2(\nu-1)$.

You will observe that the above theorems also serve to reveal in general the exact significance of your remarks concerning a quite secondary and incidental assertion contained in the quoted paper by B. Segre. It is the same author, to whom I have communicated the present considerations, who has called my attention to it. In fact, if $i'=p-2(\nu-1)+\sigma$, with $\sigma>0$, the number of moduli on which the curve depends falls *below* $2p+2\nu-5-\sigma$, in agreement with what you have verified in an example ($\sigma=1$) by a detailed and a delicate analysis.

To derive (1), I shall prove in the first place that:

1) Given an *arbitrary* curve Γ of genus p and assigned *arbitrarily* a

* *Mathematische Annalen*, Vol. 100 (1928).

g_ν^1, without fixed points, on Γ and an integer $n > 2p+1$; if furthermore H is the system, relative to the above values of n, ν, p, then the curve C_0 of H, birationally equivalent to Γ, which is being constructed by starting from the above g_ν^1 and from a *generic* g_n^1 on Γ and by setting up a projective correspondence between the groups of these series and the lines of the pencils Q, P, has the same point singularities as the generic curve C of H.

2) The $\sigma = 2n + 2p - 2$ tangents λ, drawn from P and touching C_0 outside P, have, as in the case of a generic C, a two-point contact with C_0 (and hence are distinct).

In fact, let us choose on Γ a complete g_n^{n-p} and let us consider its projective image L, which will be a curve of order n belonging to the space S_{n-p} and birationally equivalent to Γ. Since L does not possess multiple points,* a generic pencil of hyperplanes cuts out on L a g_n^1 satisfying the required conditions. In fact, the ruled surface R, generated by the lines joining the pairs of points belonging to groups of the g_ν^1 (image on L of the g_ν^1 on Γ), possesses $d = (n-1)(\nu-1) - p$ distinct generators which meet a generic S_{n-p-2}, and these are neither on multiple points of the g_ν^1, nor on sets of three or more points belonging to groups of the g_ν^1. The first statement follows from the fact that the multiple points of the g_ν^1 are finite in number; the second statement is a consequence of the fact that no generator of R is on more than two points of L, since, if L possessed a trisecant, the g_{n-1}^{n-p-1}, cut out on L by the hyperplanes through one of the three meet points, would possess a neutral pair, which is impossible, since its order $n-1$ is greater than $2p$.† If, in addition, we choose the S_{n-p-2} in such a manner, that it should not meet the virtual singular tangents of L, the g_n^1 cut out on L by the pencil on S_{n-p-2} will also satisfy the condition of possessing double points only.

If we now consider on Γ the g_n^1 homologous to the g_n^1 just constructed on L, the groups of the g_n^1, g_ν^1, which pass through a generic point of Γ, do not have other points in common, and two groups of g_n^1, g_ν^1, which have more than one point in common, have in common two points only, which are distinct. Let G, G' be two (generic) groups of g_n^1, g_ν^1, which are made up of distinct points and have no points in common, and such that the n (and the ν) groups of the g_ν^1 (and of the g_n^1), which pass through the points of G (and of G' respectively), be distinct (and also, if desired, do not

* See my *Vorlesungen über algebraische Geometrie*, Leipzig, Teubner (1921), p. 134, or also my *Trattato di geometria algebrica*, Bologna, Zanichelli (1926), p. 148.

† *Ibid.*

contain multiple points). Let us set up a projectivity π between the groups of the g_n^1 and the lines of the pencil P and a projectivity π' between the groups of g_ν^1 and the lines of the pencil Q, in such a way that to the line PQ there correspond in π and in π' the groups G, G' respectively. The curve C_0, the locus of intersections of pairs of lines on P, Q, which correspond in π, π' to groups of g_n^1, g_ν^1 passing through a variable point on Γ, satisfies, as a member of the system H of the analogous curves C, the conditions 1), 2).* From 1) and 2) it follows, that *the characteristic series of K, on C_0, is complete.* In fact, let i' be the index of speciality of the two-fold of the g_ν^1 cut out on C_0 by the lines of the pencil Q. Under this hypothesis, among the points of contact M_0 of the σ lines λ, drawn through P and touching C_0, there are $\sigma - i'$—we shall call these points N_0—which impose independent conditions on the curves D passing through the double points of C_0, so that the D's which pass through the points N_0 contain all the points M_0. We denote by Σ' the system of the curves D passing through the double points of C_0. The $\sigma - i'$ tangents relative to the points N_0 will then impose $\sigma - i''$ ($i'' \geqq i'$) conditions on the C's of H constrained to touch them, so that these C's will form a system $\bar{K}$ of dimension

$$(2n + 2\nu + p - 1) - (\sigma - i'') = 2\nu - p + 1 + i''.$$

The $\infty^{2\nu - p + i''}$ curves of $\bar{K}$, which are infinitely near to C_0, are curves D of Σ' passing through the points N_0 and consequently through all the points M_0. Hence the said curves of $\bar{K}$ cut out on C_0, outside of fixed points, groups of the characteristic series of K, which series (completed, if necessary) has the dimension $2\nu - p + i'$. Hence

$$2\nu - p + i'' \leqq 2\nu - p + i', \text{ or } i'' \leqq i',$$

and since $i'' \geqq i'$, it follows that $i'' = i'$.

Reasoning as in section 5, we deduce that the curves of $\bar{K}$ (here, and also in the analogous cases considered below, we mean of course: of an irreducible part of $\bar{K}$ which contains C_0) touch the remaining i' lines λ, and consequently $\bar{K}$ coincides with the system K, which has thus the dimension $2\nu - p + 1 + i'$. It follows that the characteristic series of K on C_0, and hence also on a generic curve of K, is complete.†

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* Cf. also in this connection my *Vorlesungen*, p. 100.

† Cf. the concluding remarks of this letter, which show that *any* curve satisfying the conditions 1), 2) belongs to only one irreducible system K.

The curves of K are birationally equivalent to C_0.* From this it follows that there exists on Γ an irreducible variety W of dimension $2(\nu-1)-p+i'$ of series g_ν^{ν}, and that the two-fold of a generic of these series has the index of speciality i'. In fact, since the (complete) characteristic series of K on a generic curve C of the system has the dimension $2\nu-p+i'$ and—consequently—the index of speciality i', it follows that i' is the index of speciality of the two-fold of the g_ν^1 cut out on the above C by the lines through Q. To this g_ν^1 on C there corresponds a g_ν^1 on Γ. Thus with every C in K there is associated a g_ν^1 on Γ, and if two curves C_1, C_2 of K are associated with the same g_ν^1 on Γ, the groups of the g_ν^1's cut out on C_1, C_2 by the lines on Q correspond to each other in a projectivity, homologous groups being those, which arise from one and the same group of the g_ν^1 on Γ in the birational correspondences between Γ, C_1 and Γ, C_2. Hence the distinct g_ν^1's on Γ, associated with the curves C of K, depend on

$$(2\nu-p+1+i')-3=2(\nu-1)-p+i' \text{ parameters.}$$

Now, considering that the g_n^1's on Γ constitute an (irreducible) variety of dimension $2(n-1)-p$ and that to every pair of series g_n^1, g_ν^1, assigned on Γ (such that g_ν^1 is in W), there correspond ∞^6 curves C birationally equivalent to C_0 and *contained, as C_0 itself, in the system H'*, we arrive immediately at the formula (1).

I now make some additional remarks which lead to the more expressive theorem relative to the case $i>0$. I prove in the first place that:

Any g_ν^1 on a curve Γ of genus p, without fixed points, belongs to a unique complete (irreducible) variety of series of the same order. Consequently two such varieties cannot have in common other than series with fixed points.

Let us denote by i the index of speciality of the two-fold of the g_ν^1 (i is not less than the greater of the two numbers 0, $p-2\nu+2$), and let C_0 be, as above, the curve constructed by means of this g_ν^1 and a generic g_n^1 on Γ ($n>2p+1$). We denote by N_0 the points of contact of $\sigma-i$ tangents λ, drawn through P, with C_0, which impose independent conditions on the curves of Σ' constrained to contain them. The curves of H, which are infinitely near to C_0 and which touch the above $\sigma-i$ lines in neighboring points of N_0, will then satisfy $\sigma-i$ independent conditions and will automatically touch the remaining lines λ. Hence these curves constitute the neighborhood I of C_0 in K. The curves of H, which touch one of the above

* See my *Vorlesungen*, p. 340, where this statement is proved in an analogous case. The same reasoning holds also here.

$\sigma - i$ lines in a point of the neighborhood of the corresponding point N_0, constitute a linear analytical branch. The $\sigma - i$ linear regions of origin C_0, which are thus obtained, possesss at C_0 independent tangent spaces. This shows that the neighborhood I consists of *only one linear branch* of origin C_0, whence the theorem follows.

As a corollary we have the following theorem:

If a g_ν^1, without fixed points, on a Γ of genus p belongs to a complete (irreducible) variety W of series of the same order, and if W has the dimension $2(\nu-1)-p+i$, then the two-fold of g_ν^1 has the index of speciality i. In fact, the index of speciality of a generic g_ν^1 is i, because the system K derived from a *fixed* generic g_n^1 $(n > 2p+1)$ and the variable g_ν^1 in W, by fixing the projective correspondence between the groups of the g_n^1 and the lines on P, is such that its characteristic series is complete. On the other hand, if C_0 is the curve which corresponds to a particular g_ν^1 without fixed points, the system of the curves C of H, which touch the lines λ, is such that its characteristic series, on C_0, is complete. But by the previous theorem that part of this system, which contains C_0, must necessarily coincide with K, and hence the theorem follows. It should be noticed that it also follows that in (1) the sign $=$ holds.

* * * * *

The proof of uniqueness, which makes use of the method of intersection of analytical branches, given by me in the "Vorlesungen," has a wider scope. It serves namely to prove, that when a continuous system K of plane curves C, of order m, is defined by certain linear conditions (such that the curves C satisfying these conditions only constitute a linear system Σ of dimension ρ) and by the condition that the C's have d nodes and σ simple contacts with a given curve Δ (which may also be reducible), then a curve C_0 of K, which possesses d nodes and σ simple contacts with Δ and *on which moreover the characteristic series of K is complete,* is the origin of a unique linear branch of the system. In fact, to say that the characteristic series of K on C_0 is complete, is the same as to say,* that the number $\sigma - i$ $(i \geqq 0)$ of conditions, imposed on the adjoint curves of order m of C_0 by the σ points of contact M_0 of C_0 with Δ, is equal to the number of conditions imposed by σ contacts with Δ on the curves of order m, possessing d ordinary double points and sufficiently near to C_0. Moreover it is possible to determine $\sigma - i$ of the points M_0—let us call them N_0—, such that the

* Cf. my Note, "Sulla teoria degl'integrali semplici di 1ª specie appartenenti ad una superficie algebrica," *Rendiconti della R. Accademia dei Lincei,* (1921), p. 297.

curves, which are near to C_0 and which touch Δ in points near to the points N_0, should necessarily touch Δ in points which are near to the remaining i points M_0. Now, in the space S_ρ of the curves of Σ we have, that the elements near to C_0, which possess a double point in the neighborhood of a double point of C_0, and the elements near to C_0, which have a contact with Δ in the neighborhood of a point N_0, constitute as many linear branches of origin C_0 and of dimension $\rho - 1$. Thus the totality of all curves of Σ near to C_0 is obtained as the intersection of $d + \sigma - i$ linear branches of origin C_0, the tangent hyperplanes of which at C_0 are linearly independent. Hence the intersection of these branches is *one* linear branch of dimension $\rho - d - \sigma + i$.

In the case in which Δ is the branch curve of a multiply covered plane, we are led to the following theorem:

Given on an algebraic surface F two continuous systems of curves of the same order and with the same number of (variable or fixed) nodes and such that the characteristic series of the systems on their respective generic curves are complete, the only curves which the two systems can have in common are those, which possess a larger number of double points, or those, on which the characteristic series is not complete.

This theorem is found in my Note of 1916,* save the last alternative, which is omitted there, since at that time the proof of the completeness of the characteristic series of a complete continuous system (Enriques) had not yet encountered the objection which I later raised against it in the quoted Note of 1921. As you know, I have proved in the same paper, by transcendental methods, that on an algebraic surface F there always exist continuous systems of curves the characteristic series of which is complete. However, it would be of great importance to prove this theorem by algebro-geometric methods. It is necessary, for this purpose, to make a thorough study of complete continuous systems of plane curves in the definition of which also occur contact conditions. These are the systems which, for the above purpose, are of most interest and for which it is not possible to affirm, at the present stage of our knowledge, that the characteristic series is complete, *even if we deal with curves possessing ordinary double points only.* The exception to Enriques' theorem in regard to systems of curves with higher singularities is of less importance, because its bearing on the geometry of the surfaces is almost insignificant. In many cases (as you have shown in an interesting example in your paper) this exception can even be removed,

* "Nuovi contributi alla teoria dei sistemi continui, etc.," *Ibid.* (1916), p. 469.

by considering the system as the limit of a system with ordinary double points only and by properly modifying the definition of a characteristic series.

I conclude with two theorems which follow immediately from the preceding considerations:

a) *On a curve* Γ *the* g_n^r*'s* $(r \geqq 1)$ *without fixed points are distributed into a finite number of continuous systems, any two of which have in common* g_n^r*'s with fixed points.*

b) *On an algebraic surface* F*, of geometric genus* $p_g > 0$*, the generic curve of any linear irreducible system* $|C|$*, which contains partially the canonical system* $|K|$ *of* F*, is of special moduli.*

In fact, if we put $C = K + D$, we will have that $|C'| = |C + K| = |D + 2K|$ is the adjoint of $|C|$. Hence the series cut out on C by $|2K|$ is special, and consequently the generic C is of special moduli, since it contains a series $|(C, K)|$ the two-fold of which is special.

ROME, FEBRUARY 27, 1930.

Grundlagen der kombinatorischen Logik.

TEIL I.

von H. B. Curry.

INHALTSÜBERSICHT.

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KAPITEL I. ALLGEMEINE GRUNDLAGEN.

A. Einleitung.

Im Anfang jeder mathematisch-logischen Untersuchung setzt man eine gewisse Menge von Kategorien voraus, die als ein Teil des irreduziblen Minimums von Kenntnis, womit man anzufangen hat, betrachtet werden sollen, Als solche Kategorien gelten gewöhnlich Aussagen und Aussagefunktionen von verschiedenen Ordnungen und Stufen. Diese Kategorien müssen weiterhin nicht nur als rein formale Begriffe vorausgesetzt werden, sondern es muss zuweilen einen Sinn haben, dass ein gegebener Gegenstand zu dieser oder jener Kategorie gehört. Mit anderen Worten, sie müssen eine inhaltliche Eigenschaft besitzen, nämlich die, dass sie Kategorien sind; und so sind sie das, was ich schon nicht-formale Grundbegriffe genannt habe.†

* Die Abschnitte *C, D, E* werden in Teil II erscheinen.

† In einer Abhandlung, "An Analysis of Logical Substitution," *American Journal of Mathematics*, Bd. 51 (Juli, 1929), S. 363-384.

Die mit diesen Kategorien verbundene vorausgesetzte Kenntnis reicht aber weit über diese Eigenschaft hinaus. In der Tat müssen wir neben den Kategorien auch gewisse Verknüpfungen annehmen—die natürlich wieder nicht-formale Grundbegriffe sind, indem sie Verfahren angeben, wonach zwei oder mehrere Gegenstände kombiniert werden können, um einen neuen Gegenstand hervorzubringen. Unter diesen Verknüpfungen gehen uns hier alle Arten von Substitutionsprozessen an. Davon muss man die folgenden Eigenschaften anfänglich erkennen: erstens diejenigen, welche uns die Möglichkeit eines Substitutionsprozesses kundgeben, wenn nur die betreffenden Gegenstände zu gewissen Kategorien gehören; zweitens die, woraus wir unter Umständen schliessen, dass zwei verschiedene Substitutionsprozesse dasselbe Resultat liefern*; drittens diejenigen, welche die Kategorie des Ergebnisses bestimmen. Diese Kategorien, Verknüpfungen und Eigenschaften bilden daher eine vorausgesetzte Lehre von beachtenswerter Verwicklung; und diese Lehre möchte ich die *Urlogik* der betreffenden Theorie nennen.

Trotz des fundamentalen Charakters dieser Urlogik für irgendeine existierende mathematische Theorie der Logik, haben doch einige der wichtigen Probleme schon in ihr ihren wesentlichen Ursprung. Ich betrachte hier die zwei folgenden:

1) Die Vereinfachung der Grundlagen. Das Wesen dieses Problems scheint mir darin zu liegen, dass man die gesamte Mathematik und Logik auf ein Minimum von Urkenntnis aufbauen will; oder genauer, dass man sie in ihre Elemente zerlegen will, damit diese Urkenntnis mit schärfster Klarheit und Deutlichkeit ausgeprägt wird. Aus diesem Grunde strebt jedermann vermutlich danach, die Anzahl der Grundbegriffe und Axiome zu vermindern; und zwar sind einige Logiker so weit in dieser Richtung gegangen, dass sie nicht davor zurückschrecken, Allgemeinheit, Vollständigkeit und sogar Exaktheit diesem Streben zu opfern.† Allein für die Logik ist die Anzahl der Grundbegriffe und Axiome dieser Art eine Betrachtung von geringerer Wichtigheit. Denn was macht es eigentlich, dass wir zwei oder drei Grundbegriffe ausschalten, wenn schon in der Urlogik selbst unendlich viele Grund-

* Vgl. Kapitel II unten; auch für einfache Beispiele den Teil II meiner oben zitierten Abhandlung.

† Z. B. müssen wir in den *Principia Mathematica*, Implikation und den bestimmten Artikel als Grundbegriffe entbehren. Die dort gegebenen Definitionen dieser Grundbegriffe führen zu den paradoxen Resultaten. In der Tat ist dort der König von Frankreich von dem König von Frankreich verschieden; und zwischen den zwei Aussagen: 1) "hätten alle Menschen drei Hände, so würde auch Bismarck drei Hände haben," 2) "hätten alle Menschen drei Hände, so wäre der Mond aus grünem Käse aufgebaut," wird kein Unterschied gemacht.

begriffe und Regeln, und zwar nicht formale, vorhanden sind? Die Vereinfachung der Grundlagen soll also mit der Vereinfachung der Urlogik beginnen.

2) Die Beseitigung der Antinomien. Um uns die Verwandtschaft von diesen mit der Urlogik zu vergegenwärtigen, betrachten wir z. B. das Russellsche Paradoxon, das in der folgenden Weise aufgefasst werden kann: F sei diejenige Eigenschaft von Eigenschaften, für welche

$$F(\phi) = \text{nicht } \phi(\phi);$$

dann ist

$$F(F) = \text{nicht } F(F).$$

Behauptet man, dass $F(F)$ eine Aussage sei, wo eine Aussage als etwas, das entweder wahr oder falsch ist, definiert wird, so hat man sofort einen Widerspruch. Aber wir können den Widerspruch allerdings dadurch vermeiden, dass wir in Abrede stellen, dass $F(F)$ zu der Kategorie der Aussagen, oder F selbst zu der der Eigenschaften gehöre. Gerade hier stossen wir auf einen Lehrsatz der Urlogik. Und zwar sind es diese Antinomien, die uns zwingen, eine entwickeltere Urlogik uns anzueignen, als sonst begreiflich wäre. Es mag also sein, dass ein tieferes Studium der Urlogik Licht in dieses dunkle Gebiet der Antinomien verbreiten wird.

Diese Umstände haben mir die Anregung gegeben, jene Urlogik mathematisch zu behandeln. D. h. genauer: eine abstrakte oder formale Theorie aufzubauen, welche auf eine sehr einfache Urlogik gegründet wird, und durch welche die Fragen, die man in der gewöhnlichen anschaulichen Urlogik zu stellen pflegt, durch symbolische Beweisführung beantwortet werden können. In dieser Abhandlung beabsichtige ich eine solche Theorie zu begründen. Weil die betreffenden Fragen einen wesentlich kombinatorischen Charakter haben, habe ich die Theorie *kombinatorische Logik* genannt.

Diese kombinatorische Logik wird wohl fähig sein als Grundlage einer abstrakten Theorie der gesamten Logik und Mathematik, einschliesslich Funktionen (Prädikaten, Relationen) von beliebig vielen Variablen, zu dienen. In der Tat bin ich der Ueberzeugung, dass eine solche Theorie durch Hinzufügen von endlich vielen formalen Grundbegriffen und Axiomen zu dem unten gegebenen Grundgerüst * gegründet werden kann. Weil in einer so eingerichteten Theorie das Grundgerüst überhaupt nur endlich viele Bestandteile hat, und weil ferner die Regeln nur ungefähr denselben Grad von Zu-

* Dieses Wort ist eine Uebersetzung der englischen Phrase "primitive frame," die ich in der oben erwähnten Abhandlung definiert habe. Es bedeutet die Gesamtheit der vorausgesetzten Grundbegriffe, Axiome und Regeln.

sammengesetztheit wie die wohlbekannte Schlussregel haben, so wird dabei ein wesentlicher Fortschritt gegen das erste oben erwähnte Problem erzielt. Was das zweite betrifft, kann ich zunächst darüber nichts aussagen.

Das in dieser Abhandlung ausgeführte Problem ist doch nur ein Teil der kombinatorischen Logik überhaupt. Dieser Teil ist die Analysis der Substitutionsprozesse, formell betrachtet und abgesehen von den Kategorien, wozu die Gegenstände gehören; d. h. ich will die zweite der drei im zweiten Paragraphen erwähnten Arten von Eigenschaften untersuchen.*

Der Weg zu dieser Analysis ist von M. Schönfinkel † angedeutet worden. Er versteht zunächst eine Funktion in einem etwas neuen Sinne, nämlich als eine Zuordnung eines Funktionswerts zu jedem Element eines Argumentbereiches, wo sowohl der Funktionswert, als auch das Argument eine Funktion sein kann. Er will dann eine Funktion von n Variablen (im gewöhnlichen Sinne) als eine Funktion (im neuen Sinne) betrachten, deren Funktionswert für das Argument x eine Funktion (im gewöhnlichen Sinne) von $(n-1)$ Variablen ist, und zwar diejenige, welche durch Einsetzung von x in die erste ‡ Leerstelle der ursprünglichen Funktion entsteht. Mit dieser Auffassung verknüpft er die folgende Bezeichnungsweise. Leerstellen gebraucht er nicht, sondern er bezeichnet Funktionen wie andere Dinge mit einfachen Buchstaben (oder Gruppen von Buchstaben). Den Funktionswert einer Funktion für ein gegebenes Argument bezeichnet er weiterhin damit, dass er das Zeichen für das Argument gleich rechts von dem für die Funktion geltenden schreibt. Zur Erläuterung dieser Ideen schreibe ich gleich unten links, einige Ableitungen von einer Funktion f, die wir im gewöhnlichen Sinne als eine Funktion von drei Variablen begreifen, und rechts dieselben Gegenstände in etwas anschaulicherer Schreibweise:

f	$f(-_1, -_2, -_3)$
fx	$f(x, -_1, -_2)$
$(fx)y$	$f(x, y, -_1)$
$((fx)y)z$	$f(x, y, z)$

Schönfinkel setzt weiter fest, dass in einem Ausdruck wie $\cdots(((fx_1)x_2)x_3)\cdots$ die die letzten Seiten mit umfassenden Klammern wegbleiben dürfen.

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* Die erste Art wird auch erledigt. In der Tat sind diese Eigenschaften nach den Betrachtungen von B 1 (unten) ganz trivial.

† *Mathematische Annalen*, Bd. 92 (1924), S. 305-316.

‡ Es ist allerdings hier vorausgesetzt, dass die Leerstellen in einer durch die Funktion selbst bestimmten Weise numeriert sind. Dieselbe Annahme liegt doch der Schönfinkelschen Darstellung zugrunde. Vgl. unten II, A.

Um nun etwas allgemeinere Ausdrücke darzustellen, setzt er einige bestimmte Funktionen fest. Diese möchte ich hier mit den Buchstaben *I, K, B, C, S* bezeichnen.* Sie sind dadurch definiert, dass für beliebige Gegenstände *x, y, z* die folgenden Definitionsregeln gelten:

$$(1)\qquad \begin{aligned} Ix &= x \\ Kxy &= x \\ Bxyz &= x(yz) \\ Cxyz &= xzy \\ Sxyz &= xz(yz). \end{aligned}$$

Vermöge dieser Funktionen ist er in der Lage, andere Ableitungen einer Funktion von drei Variabeln ohne Leerstelle darzustellen, etwa

Cfx	$f(-_1, x, -_2)$
$C(BCf)xy$	$f(y, -_1, x)$
Bfg	$f(g(-_1))$

u. s. w. In der Tat gilt nun der allgemeine Satz: Wenn irgendein logischer Ausdruck vorhanden ist, der aus gewissen Konstanten $u_1, u_2, \cdots u_m$ und numerierten freien Variablen zusammengesetzt ist, so kann er in der Schönfinkelschen Schreibweise in der Form

$$(2)\qquad Yu_1, u_2 \cdots u_m$$

wo *Y* eine Zusammensetzung von nur *I, K, B, C, S* ist, dargestellt werden. Diese Darstellung bedeutet, dass, wenn man nach (2) die Variablen in ihrer natürlichen Reihenfolge hinzufügt, und dann die Funktionen *I, K* u. s. w. durch die Definitionsregeln (1) eliminiert, der resultierende Ausdrück immer mit dem zuerst gegebenen identisch ist.—Dieser Satz ist von Schönfinkel nicht bewiesen, aber er enthält vom hier vorliegenden Gesichtspunkte aus das Wesen seiner Gedanken. Der Satz ist hier allerdings etwas roh dargelegt; genauere Sätze werden aber unten im formalen Teile streng bewiesen.

Nun macht Schönfinkel Anspruch darauf, dass er die Begriffe Variable, Aussage und Aussagefunktion aus der Logik entfernt habe. In dieser Hinsicht sind aber folgende Betrachtungen zu bemerken. Erstens: das *Y* des eben dargelegten Satzes ist doch nicht eindeutig bestimmt; z. B. sind *BCCf* und *If* in dem Sinne identisch, dass sie denselben Ausdruck darstellen. Aber diese Identität ist keineswegs aus den Definitionsregeln zu beweisen; zu diesem Behuf muss man vielmehr die Variablen tatsächlich hinzufügen. Infolge-

* Schönfinkel hat diese bzw. mit *I, C, Z, T, S* bezeichnet.

dessen sind doch die Variablen, obgleich sie in den Formeln selbst nicht erscheinen, mit allen ihren assoziierten Grundbegriffen noch unvermeidbar. Zweitens: Die Begriffe Aussage und Aussagefunktion sind auch nur scheinbar eliminiert. Denn jede Funktion hat ihren Argumentbereich, und wie definiert man diesen, wenn man jene Begriffe nicht versteht? Die allgemeinen Sätze der Logik gelten ja gewöhnlich nur, wenn die Variablen zu bestimmten Kategorien gehören. Z. B. ist der Satz von der Identität $A \rightarrow A$, nur wahr, wenn A eine Aussage ist. Weil Schönfinkel keineswegs gezeigt hat, wie die Einführung jener Grundbegriffe zu vermeiden ist, und weil er sie nicht aus anderen definieren kann, so hat er seinen Anspruch nicht gerechtfertigt. Er hat in der Tat nur eine neue und unbequemere Schreibweise gewonnen.

Jedoch führen diese Gedanken Schönfinkels zur Lösung des oben vorgelegten Problems. Denn die Funktionen B, C, u. s. w. bilden die Elemente, in die die Substitutionsprozesse sich zerlegen lassen. (Ich werde nachher genau zeigen, wie der allgemeinste solche Prozess sich aus diesen Funktionen zusammensetzen lässt). Das Wesen des Problems liegt daher in den im ersten Teile des letzten Absatzes besprochenen Identitäten. Es lässt sich wohl auf die folgende Aufgabe zurückführen: das Grundgerüst so festzustellen, dass jede dieser Identitäten rein formal bewiesen werden kann. Diese Aufgabe wird unten vollständig gelöst,* und zwar so, dass 1) die Funktionen $B, C, \cdots$ nur als rein formale Begriffe auftreten, und 2) die Variablen in den formalen Beweisen nicht zu erscheinen brauchen. Der Beweis, dass jede solche Identität aus dem Grundgerüst ableitbar ist, macht das Hauptergebnis des zweiten Kapitels aus. Dort wird auch auseinandergesetzt, wie dies die Lösung des früheren Problems liefert.

Diese Untersuchung habe ich immer in Rücksicht auf eine allgemeine Theorie der Logik durchgeführt. Dieses Ergebnis betrachte ich nur als eine Vorbereitung zu einer weiteren Theorie, die ich später fortzuführen hoffe.

B. Einige Philosophische Betrachtungen.

Bevor ich zum Aufbau der formalen Theorie übergehe, möchte ich hier einige philosophische Vorbemerkungen darlegen, die für die Theorie massgebend gewesen sind. Diese betreffen die logische Auffassung, welche der Theorie vorangeht und durch die Ausdeutung der letzteren heranreift. Mit der formalen Theorie als solcher haben sie natürlich nichts zu tun; und der Leser, der sich nicht dafür interessiert, kann sie ruhig überspringen.

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* Einige Anfänge dazu befinden sich in meiner früheren Abhandlung. Die vorliegende Behandlung ist allgemeiner und in sich vollständig; ich weise auf die frühere nur für gewisse Einzelheiten hin.

§ 1. *Sinnlose Begriffe.* In der gewöhnlichen Logik kommen gewisse Gegenstände (Begriffe, Dinge) vor, die man als sinnlos zu titulieren pflegt. Ich möchte nun gerade fragen, was dies bedeutet. Man darf wohl behaupten, dass ein Wort oder ein Zeichen in bezug auf eine Sprache noch nicht definiert ist. Z. B. ist das Wort "cow" auf deutsch sinnlos, doch gelegentlich anf englisch sinnvoll. Aber in den vorhandenen Fällen sind es nicht Worte, sondern Begriffe, die sinnlos sein sollen. Z. B. behaupten Whitehead und Russell (und auch andere), dass $\phi(\phi)$ für jedes ϕ bedeutungslos sei. Zu sagen, dass diese Behauptung nur das Zeichen $\phi(\phi)$ betrifft, heisst, die Frage zu umgehen; weil es doch erstens ein gedachtes Etwas gibt, das das Zeichen $\phi(\phi)$ den Konventionen gemäss bedeuten mag, und zweitens, weil die Gründe,—nämlich das "Vicious Circle Principle" u. s. w. —, woraus die Behauptung entsteht, mit dem Zeichen gar nichts zu tun haben. Die Sinnlosigkeit scheint doch einen sachlichen Inhalt zu haben, aber welchen, lässt sich nicht sagen.

Zunächst gibt es aber einen Sinn, worin alles Denkbare eine Bedeutung hat, nämlich als Begriff. Hier ist ein Begriff als irgendetwas zu verstehen, das identifiziert und von anderen Dingen unterschieden zu werden vermag.* Dann ist es allerdings unbedingt Unsinn zu sagen, dass etwas nicht als Begriff existiere; denn bevor man einen solchen Satz verstehen konnte, muss man sich die Sache als Begriff schon vorgestellt haben. Sogar die "sinnlosen" Gegenstände sind also Begriffe und haben als solche eine Bedeutung.

Unter den Begriffen kommen aber einige vor, die "in sich widerspruchsvoll" sind. Solche sind das oben erwähnte $F(F)$, die grösste Kardinalzahl, die kleinste undefinierbare Ordinalzahl, das kreisförmige Quadrat u. s. w. Diese Begriffe sollen wegen der Widersprüche bedeutungslos sein. Aber die Widersprüche liegen nicht in den Begriffen selbst, sondern in den Eigenschaften, die man ihnen zuordnen möchte: z. B. führt das oben erwähnte $F(F)$ zu einem Widerspruch nur dann, wenn man behauptet, dass es eine Aussage sei; die grösste Kardinalzahl nur dadurch, dass sie wirklich eine Kardinalzahl sei u. s. w.

Betrachten wir nun die sinnlosen Gegenstände überhaupt. Bezieht sich nicht derselbe Gedanke auch auf sie? Ja, die Sinnlosigkeit dieser Begriffe besteht nur darin, dass es Eigenschaften gibt, die sie nicht besitzen. Und zwar, dürfte ich genauer sagen, dass sie den gewöhnlichen Kategorien nicht

* Es ist wohl bemerkenswert, dass ein Begriff nach dieser Definition ein Gegenstand und nicht ein Prozess des Denkens ist. Beispiele von Begriffen sind etwa Bismarck, Göttingen, Tier, Regenschirm, rot, Temperatur, Materie, Substanz, Kausalität, Etwas, Funktion, der König von Frankreich, die grösste Kardinalzahl u. s. w.

angehören. Diese Kategorien sind in der Tat als etwa inhaltliche Grundbegriffe vorausgesetzt, und nichts wird betrachtet ausser was dazu gehört. Natürlich müssen Begriffe, die mit dem Wesen dieser Kategorien unverträgliche Eigenschaften besitzen, als "sinnlos" aus der Theorie ausgeschlossen werden. Aber gerade in diesem Ausschliessen besteht ein Mangel. Die Aufgabe der Logik ist die Erklärung des Denkens; wenn es von der Erklärung ausgeschlossenes Denken gibt, so ist sie fehlerhaft. Weiterhin sind es genau diese sinnlosen Begriffe, die zu Widersprüchen führen; wenn man sie ausschliesst, so kann man wohl die Antinomien vermeiden, aber nie erklären. Dass etwas ein Begriff ist, ist das einzige Erfordernis, damit man das Ding in der Logik behandeln könne.

Die Kategorie Begriff—oder, wie ich sie nachher nennen werde, um gewisse Nebenbedeutungen zu vermeiden, Etwas,—ist daher die grundlegende Kategorie der Logik überhaupt. Diese Kategorie ist ein Begriff, und seine blosse Betrachtung ist doch in sich widerspruchsfrei. Aus diesen Gründen habe ich sie als Grundkategorie der Theorie vorausgesetzt. Daraus folgt eine wichtige Konsequenz: ich brauche mich nicht mehr, mindestens insofern als es die Einführung neuer Gegenstände in die Theorie betrifft, um die Definitionsbereiche von Funktionen zu kümmern. Wenn ich z. B. den Schönfinkelschen Funktionsbegriff als Grundverknüpfung — und Schöfinkel hat schon gezeigt, dass dies allein notwendig ist—, voraussetze, so ist die Zusammensetzung von irgend zwei Begriffen, etwa f und x zu fx, wieder ein Begriff. Z. B. ist die Zusammensetzung des Königs von Frankreich mit der Aussage "der Mond ist aus grünem Käse aufgebaut," ein Begriff, weil sie identifiziert und von anderen Dingen unterschieden zu werden vermag. Unter den so hergestellten Begriffen werden einige "sinnvoll," andere "sinnlos" sein; die Hauptaufgabe der kombinatorischen Logik ist wohl, diese zwei Arten zu unterscheiden.*

§ 2. *Der Vorrang des Aussagenkalküls.* In den heutigen logischen Theorien bildet der sog. Aussagenkalkül den Grundbestandteil. Ich möchte hier die vielleicht triviale Bemerkung darlegen, dass dieser Vorgang nicht notwendig ist. Natürlich beginnt die Logik mit Sätzen, die wir fähig sein müssen, zu verstehen. Aber daraus folgt ebensowenig, dass man mit der allgemeinen Theorie von Aussagen anfangen muss, als aus der Tatsache, dass wir es im Aussagenkalkül mit gewissen Aussagefunktionen zu tun haben, folgt, dass wir mit Aussagefunktionen beginnen müssen. Weiterhin sind

* Dabei wird die erste Art von Eigenschaften die ich in I A erwähnt habe, auf die Dritte zurückgeführt.

die Begriffe "Aussage" und "Behauptung" verschieden; dieser ist mit der Wahrheit eng verbunden, jener eine Kategorie von bloss betrachteten Etwasen. Was wir im Anfang verstehen müssen, ist erstens, dass die Sätze Behauptungen sind, und zweitens, dass sie sich vermöge der Regeln zu neuen Behauptungen umgestalten lassen. Was wir mit solchen einfachen Anfängen machen können, zeigt die hier gegebene Theorie an.

Der Begriff von Behauptung muss natürlich als Grundbegriff angenommen werden und zwar als ein solcher, der in der formalen Theorie einem nicht formalen Begriffe entspricht. Diesen nicht formalen Begriff habe ich "*Formel*" genannt.

§ 3. *Unbeschränkte Universalen.* In Nummer 1 habe ich behauptet, dass der Begriff "Etwas" als grundlegender Begriff angenommen werden darf. Ich behaupte nun ferner, dass es Eigenschaften gibt, die für jedes Etwas gelten. Gegen die Möglichkeit solcher Behauptungen steht aber das wohlbekannte Verbot der nicht prädikativen Begriffsbildungen. Dieses Verbot hat im allgemeinen einen etwa pragmatischen Charakter, und lässt sich folglich nur dadurch widerlegen, dass man wirklich eine dem Verbot widersprechende Theorie aufbauen kann. In den "Principia Mathematica" wird aber dieses Prinzip in der Form des "Vicious Circle Principle," a priori verteidigt. Das dortige Argument ist etwa dies: man könnte nichts über alle Aussagen behaupten, weil dadurch neue Aussagen erschaffen würden, und also es keine bestimmte Gesamtheit von Aussagen gäbe, die den Wirkungsbereich der behaupteten Eigenschaft bildete. Ebensogut aber ist dieses Argument: man könnte nichts über alle Apfelsinen behaupten, weil jedes Jahr neue Apfelsinen erschaffen werden, und es also keine bestimmte Gesamtheit von Apfelsinen gäbe u. s. w. u. s. w. In der Tat begreifen wir allgemeine Urteile nicht in extenso, sondern in intenso—es gibt etwas im Wesen der betreffenden Eigenschaft, wovon wir schliessen können, dass sie für jedes Etwas gilt. Dieser Schluss setzt doch keine Gesamtheit voraus. Es fehlt daher diesem Einwand gegen die Möglichkeit unbeschränkter Universalen ein theoretischer Grund. Infolgedessen mache ich die einfachere und naturgemässere Voraussetzung, dass es solche Universalen gibt; insbesondere, dass solche Eigenschaften wie das reflexive Gesetz der Gleichheit für jedes Etwas gelten.

Wenn wir aber diese Annahme machen, so werden die Grundregeln der Theorie erheblich einfacher. Denn diese Regeln sind Universalen; wenn wir sie als unbeschränkte Universalen auffassen können, so brauchen wir keine Unterscheidungen zwischen verschiedenen Arten von Begriffen inbezug auf die Regeln zu machen. Eine ähnliche Vereinfachung betrifft das ganze Grundgerüst.

§ 4. *Theorie und Metatheorie.* Die letzte Betrachtung hat die Methode der Untersuchung zum Gegenstand. Weil die im Anfang vorausgesetzten Regeln sehr einfach sind, ist eine sehr lange Darlegung erforderlich, um diese Entwicklung ausführlich zu vervollständigen. Wir interessieren uns aber nur für die Möglichkeit einer solchen Darlegung. Ich gebe demgemäss, statt die Formeln eine nach der anderen auszuschreiben, hier eine Reihe von Sätzen an, worin ich zeige, dass bestimmte Arten von Etwasen immer Formeln sind. Diese Sätze gehören eigentlich zu der Metamathematik im Hilbertschen Sinne, die mit dieser Mathematik verbunden ist. In den Beweisen dieser Sätze benutze ich nur die einfachen direkten Beweismethoden, und zwar werden die Beweise so aufgestellt, dass sie selbst, natürlich mit Benutzung der Nachweisungen, einen Prozess ergeben, wodurch der Beweis irgendeines in sie eingeschlossenen Falles ausführlich Formel nach Formel auseinandergesetzt werden kann. Unter die Methoden kommen allerdings die vollständige Induktion und andere Eigenschaften der ganzen Zahlen vor, aber dies bedeutet nicht mehr, als dass wir die ganzen Zahlen als Zeichen benutzen, und dass für einige Etwasen, die mit höheren Zahlen bezeichnet werden, jene Beweisprozesse wiederholt werden sollen. Für alle besonderen Fälle hat man eine endliche direkte Schlussfolge.

C. Das Grundgerüst.

Ich gehe jetzt zur formalen Theorie über. Die erste Aufgabe ist natürlich die Darlegung des Grundgerüsts und die Erklärung der in der Ausführung benutzten Festsetzungen.

§ 1. *Vorbereitende Erklärungen.* Die vorliegende Theorie wird vom abstrakten Gesichtspunkte aus durchgeführt; d. h. als eine Lehre über abstrakte Begriffe, die durch das Grundgerüst selbst definiert werden.* Es ist natürlich wichtig darin die formalen Entwicklungen selbst und die inhaltlichen Ueberlegungen darüber zu unterscheiden; aber ausserdem habe ich noch die Unterscheidung zwischen der Theorie selbst und ihrer Symbolik festgehalten. Jedoch kann der Leser, der einen anderen Gesichtspunkt vorzieht, wohl die Theorie durch kleine Aenderungen nach seinem eigenen Gesichtspunkt umgestalten. Z. B. ist das, was ich oben über die Endlichkeit der Anzahl von Axiomen ausgesagt habe, mit dem Gesichtspunkt verbunden, dass eine Definition nichts wesentlich Neues in die Theorie hereinbringt, und also von

* Ueber das Wesen einer abstrakten Theorie, vgl. meine oben zitierte Abhandlung.

einem Axiom verschieden ist; aber sonst kann man immer einen Satz desselben Inhalts mit etwas anderen Worten aussprechen.

In den Ausführungen brauche ich zwei Arten von Zeichen: erstens die *formalen Zeichen,* welche die Bestandteile der formalen Theorie selbst bedeuten; zweitens die *inhaltlichen Zeichen,* welche eine Bedeutung inbezug auf die inhaltlichen Überlegungen haben. Mit diesen zwei Zeichenarten sind zwei Arten von Definitionen zu betrachten; um diese zu unterscheiden, beschränke ich nunmehr das Wort *Definition* auf die Erklärungen der formalen Zeichen, während die Erklärungen der inhaltlichen Zeichen (bzw. Wörter, die in einem technischen Sinn gebraucht werden), *Festsetzungen* heissen sollen. Die Definitionen und Festsetzungen, die für die ganze Abhandlung beibehalten werden sollen, werden hervorgehoben und als solche ausgezeichnet; neben ihnen mache ich in den einzelnen Beweisen und Erläuterungen Gebrauch von solchen, die nur für den betreffenden Zusammenhang gelten sollen.

Die Definitionen werden weiterhin nun folgendermassen festgestellt. Erstens werden die Zeichen, die in der Darstellung des Grundgerüsts selbst mit gewissen Bestandteilen verbunden sind, dabei definiert. Jede spätere Definition ist dann eine Bestimmung, dass ein nicht schon in demselben Kontext definiertes Zeichen dasselbe wie ein anderes schon definiertes bedeuten soli. Die Identitätsrelation, die in diesem "dasselbe" schon vorliegt, kann, wenn man will, formalisiert werden. Definitionen dürfen sich natürlich nicht nur auf Zeichen im engeren Sinne, sondern auch auf Zeichenverknüpfungen beziehen (vgl. unten Def. 1 und 2).

In der folgenden Darlegung des Grundgerüsts gebe ich neben den zur abstrakten Theorie selbst erforderlichen Eigenschaften auch Andeutungen über die logische Interpretation des betreffenden Begriffs, Regel u. s. w. an. Diese Andeutungen sind aber in Klammern gesetzt.

§ 2. *Die nicht-formalen Grundbegriffe.* Die folgenden Begriffe seien vorausgesetzt:

a. *Etwas,* eine Kategorie (Ausdeutung, der oben in B 1 diskutierte Begriff Etwas). Etwase werden durch Buchstaben oder ähnliche Zeichen bezeichnet.

b. *Formel,* eine Kategorie (Ausdeutung: Behauptung).

c. *Anwendung,* eine dyadische Verknüpfung, d. h. eine Zuordnung, wodurch zu jedem geordneten Paar von Etwasen ein eindeutig bestimmtes Drittes zugeordnet wird. Wenn die zwei Etwase durch X bzw. Y bezeichnet sind, dann wird das zugeordnete Dritte die

Anwendung von X auf Y genannt und durch (XY) bezeichnet. (Ausdeutung: der Schönfinkelsche Funktionsbegriff, erweitert wie in B 1 oben).

§ 3. *Die formalen Grundbegriffe.* Von diesen Begriffen wird nichts vorausgesetzt, ausser dass sie Etwase sind. Ich schreibe also hier nur ihre Zeichen und Ausdeutungen:

B (Die Schönfinkelsche Zusammensetzungsfunktion).
C (Die Schönfinkelsche Vertauschungsfunktion).
W (eine Funktion derselben Art wie die vorhergehende, durch die Regel W [s. unten] definiert. Wir dürften sie die Verdoppelungsfunktion nennen).
K (Die Schönfinkelsche Konstanzfunktion).
Q (Gleichheit, d. h. logische, nicht, symbolische Identität).
Π (Allzeichen).
P (Implikation).
Λ (Konjunktion—das Und—Vgl. unten, Reg. Λ).

Die vorhergehenden Begriffe sind die einzigen, die in dieser Abhandlung erwähnt sind; für die weitere Durchführung der Logik brauchen wir natürlich gewisse andere, z. B. Aussage, Negation, Funktion, Seinzeichen (wenn nicht definiert) u. s. w.

§ 4. *Symbolische Festsetzungen.*

Festsetzung 1. Wenn X ein Etwas ist, so bezeichne ich mit $\vdash X$ den Satz, dass X eine Formel ist, und zwar sowohl wenn dieser Satz behauptet, festgesetzt oder bloss betrachtet ist. Weiterhin dürfen dann die äusseren Klammern von X, wenn es solche gibt, wegbleiben.

Festsetzung 2. Wenn X und Y Etwase sind, so bezeichne ich mit $X \equiv Y$ den Satz, dass X und Y dasselbe Etwas sind, d. h. dass X und Y als Zeichen betrachtet, dasselbe Etwas bedeuten. Unter diesen Umständen werde ich auch manchmal sagen, dass X und Y *identisch* sind. Die äusseren Klammern von X und Y dürfen auch hier wegbleiben.

Def. 1. Immer wenn $X_1, X_2, \cdots, X_n$ Etwase sind, gilt

$$(X_1X_2X_3\cdots X_n) \equiv (\cdots(((X_1X_2)X_3)\cdots)X_n).$$

Def. 2. Immer wenn X und Y Etwase sind, gilt

$$(X = Y) \equiv (QXY).$$

Links dürfen weiter die äusseren Klammern von X und Y wegbleiben, solange nur X und Y nicht selbst von der Form $(U = V)$ sind.

Def. 3. $I \equiv (WK)$.

Festsetzung 3. Die Relation zwischen zwei Etwasen X und Y, welche durch $\vdash X = Y$ ausgedrückt wird, heisst *Gleichheit*, und unter diesen Umständen sage ich, dass X und Y gleich sind.

§ 5. *Axiome.* Die hier gegebenen Axiome reichen nur für diese Abhandlung aus. Die kombinatorischen Axiome werden unten (s. II B 5)* in viel übersichtlicherer Form wiedergegeben.

a. *Axiom der Identität.*

Ax. Q. $\vdash \Pi(W(CQ))$ (Reflexives Gesetz, s. unten, Abschn. D).

b. *Kombinatorische Axiome.*

Ax. B. $\vdash C(BB(BBB))B = B(BB)B.$
Ax. C. $\vdash C(BB(BBB))C = B(BC)(BBB).$
Ax. W. $\vdash C(BBB)W = B(BW)(BBB).$
Ax. K. $\vdash C(BBB)K = B(BK)I.$
Ax. I_1. $\vdash CBI = B(BI)I.$†
Ax. (BC). $\vdash BBC = B(B(BC)C)(BB).$
Ax. (BW). $\vdash BBW = B(B(B(BW)W)(BC))B(BB))B.$
Ax. (BK). $\vdash BBK = BKK.$
Ax. $(CC)_1$. $\vdash BCC = B(BI).$
Ax. $(CC)_2$. $\vdash B(B(BC)C)(BC) = B(BC(BC))C.$
Ax. (CW). $\vdash BCW = B(B(BW)C)(BC).$
Ax. (CK). $\vdash BCK = BK.$
Ax. (WC). $\vdash BWC = W.$
Ax. (WW). $\vdash BWW = BW(BW).$
Ax. (WK). $\vdash BWK = BI.$
Ax. I_2. $\vdash BI = I.$

§ 6. *Regeln.* Die folgenden Regeln sind vermutlich für die gesamte Mathematik und Logik hinreichend. Die zwei letzteren werden nur in einem späteren Stadium benutzt, aber sie sind hier der Vollständigkeit halber gegeben.

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* s. d. für Bemerkungen über die Bedeutung dieser Axiome.

† Dieses Axiom kann aus den anderen bewiesen werden. s. unten. II D 2, Satz 6.

Reg. E. Wenn X und Y Etwase sind, dann ist immer (XY) ein Etwas. (Ueber die Ausdeutung dieser Regel vgl. oben, B 1. Sie ist so einfach, dass sie nie wieder erwähnt wird, sondern nur implizit auftritt).

Reg. Q_1. Wenn X und Y Etwase sind, und 1) $\vdash X$, 2) $\vdash QXY$ (bzw. $\vdash X = Y$), dann $\vdash Y$.

Reg. Q_2. Wenn X, Y, Z Etwase sind, und $\vdash QXY$ (bzw. $\vdash X = Y$),

dann $\vdash Q(ZX)(ZY)$ (d. h. $\vdash ZX = ZY$).

Reg. Π. Wenn X und Y Etwase sind, und $\vdash \Pi X$, dann $\vdash XY$. (Die Ausdeutung dieser Regel ist das sog. Prinzip von Aristoteles).

Reg. B. Wenn X, Y, Z Etwase sind, dann $\vdash BXYZ = X(YZ)$.

Reg. C. Wenn X, Y, Z Etwase sind, dann $\vdash CXYZ = XZY$.

Reg. W. Wenn X, Y Etwase sind, dann $\vdash WXY = XYY$.

Reg. K. Wenn X, Y Etwase sind, dann $\vdash KXY = X$.

Reg. P. Wenn X, Y Etwase sind, sodass 1) $\vdash X$, 2) $\vdash PXY$, dann $\vdash Y$. (Dies ist die wohlbekannte Schlussregel).

Reg. Λ. Wenn X, Y Etwase sind, sodass 1) $\vdash X$, 2) $\vdash Y$, dann $\vdash \Lambda XY$. (Diese Regel kann natürlich vermieden werden, wenn in den formalen Entwicklungen der Theorie geschlossen werden kann, dass für beliebige X, Y

$$\vdash PX(PY(\Lambda XY)),$$

d. h. in gewöhnlicherer Schreibweise,

$$X \rightarrow (Y \rightarrow (X \,\&\, Y)).$$

Aber diese Aussage ist mit der logischen Bedeutung der Implikation unverträglich. Wenn wir diese logische Implikation eintreten lassen wollen, so müssen wir P und Λ als unabhängige Grundbegriffe, und die beiden Regeln P und Λ voraussetzen. [Vgl. Lewis, C. J.—*A Survey of Symbolic Logic.* Berkeley 1918, Chap. V]).

D. Die Eigenschaften der Gleichheit.

Satz 1. *Wenn X ein Etwas ist, dann* $\vdash QXX$.

Beweis:	$\vdash \Pi(W(CQ))$	(Ax. Q).
$\therefore$	$\vdash W(CQ)X$	(Reg. Π, Hp.).
Aber	$\vdash Q(W(CQ)X)(CQXX)$	(Reg. W).
$\therefore$	$\vdash CQXX$	(Reg. Q_1).
Aber	$\vdash Q(CQXX)(QXX)$	(Reg. C).
$\therefore$	$\vdash QXX$ w. z. b. w.	(Reg. Q_1).

SATZ 2. *Wenn X und Y Etwase sind, und* $\vdash QXY$, *dann* $\vdash QYX$

Beweis: $\vdash QXY$ (Hp.)
$\therefore$ $\vdash Q(CQXX)(CQXY)$ (Reg. Q_2, $Z \equiv CQX$).
Aber $\vdash CQXX$ (vorvorletzte Formel im Beweis von Satz 1).
$\therefore$ $\vdash CQXY.$ (Reg. Q).
$\therefore$ $\vdash QYX.$ w. z. b. w. (Reg. C und Q_1).

SATZ 3. *Wenn X, Y, Z Etwase sind, und* 1) $\vdash QXY$, 2) $\vdash QYZ$, *dann* $\vdash QXZ$.

Beweis: $\vdash QYZ$ (Hp. 2).
$\vdash Q(QXY)(QXZ)$ (Reg. Q_2).
$\therefore$ $\vdash QXZ$ w. z. b. w. (Hp. 1 und Reg. Q_1).

SATZ 4. *Wenn X, Y, Z Etwase sind, und* $\vdash QXY$, *dann* $\vdash Q(XZ)(YZ)$.

Beweis:
$\vdash Q(C(B(WK))ZX)(C(B(WK))ZY)$ (Hp. und Reg. Q_2).
$\vdash Q(C(B(WK))ZX)(B(WK)XZ)$ (Reg. C).
$\vdash Q(B(WK)XZ)(WK(XZ))$ (Reg. B).
$\vdash Q(WK(XZ))(K(XZ)(XZ))$ (Reg. W).
$\vdash Q(K(XZ)(XZ))(XZ)$ (Reg. K).

aus den letzten vier Formeln und Satz 3, $\vdash Q(C(B(WK))ZX)(XZ)$. In ähnlicher Weise folgt dass $\vdash Q(C(B(WK)ZY))(YZ)$.

Aus der ersten und den zwei letzten Formeln und den Sätzen 2 und 3.
$\vdash Q(XZ)(YZ).$ w. z. b. w.

SATZ 5. *Wenn X und Y Etwase sind, und* $X \equiv Y$, *dann* $\vdash QXY$.

Beweis: Aus Satz 1 $\vdash QXX$.
Weil Y dasselbe wie X bedeutet, so folgt $\vdash QXY$. w. z. b. w.

SATZ 6. *Gleichheit ist eine reflexive, symmetrische und transitive Relation, und zwar derart, dass aus* $\vdash X = Y$ *für ein beliebiges Etwas Z folgt*

$$\vdash ZX = ZY, \text{ und } \vdash XZ = YZ,$$

und weiterhin aus $\vdash X$ *und* $\vdash X = Y$, *folgt* $\vdash Y$.

Beweis: Dieser Satz ist nur eine Zusammenfassung von Reg. Q_1 und Q_2 und Sätzen 1-4 inbezug auf C 4. Def. 2 und der Bedeutung von $\equiv$. Er lässt

sich aber auch, abgesehen von der Bedeutung von $\equiv$, nur mit Benutzung von Satz 5 ableiten. Z. B. beweise ich die letzte Behauptung.

Angenommen	$\vdash X = Y,$	
dann	$\vdash QXY$	(C 4. Def. 2, Satz 5, Reg. Q_1),
daher aus	$\vdash X,$	
folgt	$\vdash Y$	(Reg. Q_1).

SATZ 7. *$\mathfrak{X}$ sei ein Etwas, das aus gegebenen Etwasen $X_1, X_2, \cdots, X_n$ (unter anderen) durch Anwendung entsteht, und $\mathfrak{Y}$ sei der Ausdruck, welcher dann entsteht, wenn man $X_1, X_2 \cdots X_n$ in $\mathfrak{X}$ durch Etwase $Y_1, Y_2 \cdots Y_n$ ersetzt. Ferner sei für alle $i = 1, 2 \cdots n$ entweder $\vdash X_i = Y_i$ oder $\vdash Y_i = X_i$. Dann $\vdash \mathfrak{X} = \mathfrak{Y}$.*

Beweis: Wir können annehmen, dass X_i nur einmal in $\mathfrak{X}$ vorkommt, weil der Fall, dass einige X_i nochmals vorkommen, sich auf den Fall, dass sie alle nur einmal vorkommen, durch Vergrösserung des n zurückführen lässt (die X_i brauchen nicht alle verschieden zu sein). Weiterhin können wir uns auf den Fall $n = 1$ beschränken, weil es sonst immer eine Reihe von Ausdrücken $\mathfrak{X}_0 \equiv \mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2, \cdots \mathfrak{X}_n \equiv \mathfrak{Y}$ derart gibt, dass $\mathfrak{X}_{i+1}$ sich aus $\mathfrak{X}_i$ durch Einsetzung von Y_i statt X_i erzeugt, und wenn wir dann bewiesen haben, dass

$$\vdash \mathfrak{X}_i = \mathfrak{X}_{i+1},$$

so folgt	$\vdash \mathfrak{X}_0 = \mathfrak{X}_n$	(Satz 6),
also	$\vdash \mathfrak{X} = \mathfrak{Y}$	(Satz 5, 6).

Infolgedessen nehmen wir an, dass $\mathfrak{Y}$ sich aus $\mathfrak{X}$ durch Einsetzung von Y statt X erzeugt. Dann gibt es zwei Reihen von Etwasen $X_1, X_2 \cdots X_m$, $Y_1, Y_2, \cdots Y_m$ derart, dass

$$X_1 \equiv X, \qquad Y_1 \equiv Y,$$
$$X_{i+1} \equiv X_i Z_i, \text{ oder } X_{i+1} \equiv Z_i X_i,$$
und
$$Y_{i+1} \equiv Y_i Z_i, \text{ bzw. } Y_{i+1} \equiv Z_i Y_i,$$
$$X_m \equiv \mathfrak{X}, \qquad Y_m \equiv \mathfrak{Y}.$$

Dann haben wir zuerst $\vdash X_1 = Y_1$ (Hp., Satz 5,—vielleicht Satz 3),
und weiter aus $\vdash X_i = Y_i$ folgt $\vdash X_{i+1} = Y_{i+1}$ (Satz 6).
Daher durch Induktion $\vdash X_m = Y_m$,
daher $\vdash \mathfrak{X} = \mathfrak{Y}.$ w. z. b. w.

SATZ 8. *Die zwei Sätze* 1) $\mathfrak{X} \equiv \mathfrak{Y}$ *und* 2): *"Aus den vorangegebenen Sätzen der Form $X_i \equiv Y_i$ kann der Satz $\vdash \mathfrak{X} = \mathfrak{Y}$ allein mit Benutzung von Satz 5 und die Eigenschaften der Gleichheit bediesen werden," sind äquivalent.*

Beweis: Der allgemeinste Satz der Form $\mathfrak{X} \equiv \mathfrak{Y}$, den wir nach der Bedeutung von $\equiv$ aus den vorangegebenen schliessen können, ist ein Spezialfall von Satz 7 mit Gleichheit durch Identität ersetzt. Aber Satz 7 gibt auch den allgemeinsten Satz der Form $\vdash \mathfrak{X} = \mathfrak{Y}$ welcher aus $\vdash X_i = Y_i$ durch Benutzung der Eigenschaften der Gleichheit bewiesen werden kann. Damit wird die Behauptung bewiesen.

Festsetzung 1. Unter

$$\vdash X_1 = X_2 = X_3 \cdots = X_n$$

verstehen wir

$$\vdash X_1 = X_2 \text{ und } \vdash X_2 = X_3 \text{ und } \cdots \text{ und } \vdash X_{n-1} = X_n,$$

(woraus insbesondere folgt $\vdash X_1 = X_n$. Die Eigenschaften werden der Gleichheit hiernach im allgemeinen nicht besonders erwähnt).

SATZ 9. *Wenn X ein Etwas ist*

$$\vdash IX = X.$$

Beweis:

$\vdash IX = WKX$	(C4. Def. 3, Satz 4),
$= KXX$	(Regel W),
$= X$	(Regel K).

Bemerkung: Anstatt B, C, W und K hat Schönfinkel K und S als Grundbegriffe gewählt. Dazu gehören die Regeln

$\vdash KXY = X$	für beliebige Etwase X, Y;
$\vdash SXYZ = XZ(YZ)$	für beliebige Etwase X, Y, Z,

woraus sich B, C, W und I wie folgt definieren lassen:

$$B \equiv S(KS)K, \quad C \equiv S(BBS)(KK), \quad W \equiv SS(SK), \quad I \equiv SKK.$$

In dieser Weise kann man wohl die Anzahl der Grundbegriffe und Regeln vermindern. Der Beweis der Eigenschaften der Gleichheit ist aber etwas schwieriger, und man muss vermutlich den obigen Satz 4 als Grundregel voraussetzen.

KAPITEL II. DIE LEHRE DER KOMBINATOREN.

A. EINLEITUNG.

Festsetzung 1. Unter einer *Kombination* von Etwasen X_1, $X_2 \cdot \cdot \cdot X_n$ verstehen wir ein Etwas, das sich aus $X_1 \cdot \cdot \cdot X_n$ durch Anwendung aufbauen lässt. Genauer definieren wir so:

1) Jedes X_i $(i = 1, 2, \cdot \cdot \cdot, n)$ ist eine Kombination von $X_1 \cdot \cdot \cdot X_n$.
2) Wenn X und Y Kombinationen von $X_1, \cdot \cdot \cdot X_n$ sind, so ist auch (XY) eine solche.

Festsetzung 2. Unter einem *Kombinator* verstehen wir eine Kombination von B, C, W und K.

In diesem Kapital untersuchen wir die Kombinatoren als solche. Das Hauptergebnis betrifft aber eine Verwandtschaft mit den Substitutionsprozessen. Es wird gezeigt, dass alle die in der gewöhnlichen Logik durch freie Variablen angedeuteten Verknüpfungen sich durch Kombinatoren definieren lassen, und zwar so, dass alle die Eigenschaften, die sie dort haben, auch aus unserem Grundgerüst ableitbar sind.

Um dieses Problem schärfer zu formulieren, betrachten wir diese Operationen und ihre Ausdrucksweise näher. Es soll genau erklärt werden, was sie sind und welche Eigenschaften in Betracht kommen.

Zunächst dienen natürlich die Variablen nur dazu, die Leerstelle in den verschiedenen Funktionen zu unterscheiden. Infolgedessen darf ich für sie eine besondere Bezeichnungsweise fordern, und zwar die folgende: Man beginnt mit gewissen Grundkonstanten und Grundfunktionen. In diesen sollen die Leerstelle mit den Zeichen x_1, $x_2 \cdot \cdot \cdot x_n$ konsekutiv (d. h. ohne Auslassungen)* numeriert werden, und zwar in einer Anordnung, die am Anfang beliebig zu wählen ist, aber danach festbleiben soll. Dann werden die folgenden Operationen für die Erschaffung neuer Funktionen und Aussagen erlaubt.

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* [Oder genauer, wenn x_e als Argument vorkommt, so soll auch x_k für $k < e$ vorkommen. (P. Bernays)].

† In der gewöhnlichen Theorie sagt man zuweilen, dass ese zwischen $\phi(x, y)$ und $\phi(y, x)$ keinen Unterschied gebe, weil es nicht darauf ankomme, wie man die Variablen bezeichne. Aber wenn man eine Funktion $\phi(x, y)$ hat, so kann man andere Funktionen $\psi(x, y)$, $\chi(x)$ definieren, sodass

$$\psi(x, y) \equiv_{xy} \phi(y, x) \qquad \chi(x) \equiv_{x} \phi(x, x).$$

Dann ist der Inhalt meiner Forderung dieser: wenn wir z. B. ϕ mit $\phi(x_1, x_2)$ bezeichnen, dann werden ψ und χ mit $\phi(x_2, x_1)$ bzw. $\phi(x_1, x_1)$ bezeichnet. Natürlich haben wir hier im allgemeinen drei verschiedene Funktionen.

1) ***Umwandlung*** einer Funktion in eine zweite, die sich von der ersten nur darin unterscheidet, dass die Variablen anders numeriert sind. Diese neue Numerierung soll die obigen Bedingungen der Konsekutivität erfüllen, doch ist es erlaubt, Leerstellen, die ursprünglich verschiedene Zeichen trugen, dasselbe Zeichen zu geben, aber nicht umgekehrt. Z. B. aus $\phi(x_1, x_2)$ werden $\phi(x_2, x_1)$ und $\phi(x_1, x_1)$ durch Umwandlungen geschaffen usw. Die so umgeformten Funktionen, die verschiedene Bezeichnungen haben, sind als ganz voneinander und von der Ursprünglichen verschiedene Funktionen anzusehen.†

2) Einsetzung einer Konstanten a für eine Variable x_k in einer Funktion von n Variablen, wo $n \geqq k$. Dadurch wird eine Funktion von $n-1$ Variablen (bzw., wenn $n=1$ ist, eine Konstante) geformt. Dies soll in der folgenden Weise bezeichnet werden: für x_k in der ursprünglichen Funktion soll immer a auftreten, für x_i mit $i > k$ soll x_{i-1} auftreten, während die x_i mit $i < k$ unverändert bleiben sollen.

3) Zusammensetzung einer Funktion von n Variablen mit einer von m Variablen, durch Einsetzung von dieser *als Funktion von m Variablen* für x_k ($k \leqq n$) in jene. Die so gestaltete Funktion soll in der folgenden Weise bezeichnet werden: Zunächst setzt man in die zweite Funktion y_i für x_i ein, dann setzt man den so umgeformten Ausdruck als ein Ganzes in die erste Funktion in alle die Stellen ein, wo x_k da erscheint, und endlich formt man den resultierenden Ausdruck so um, dass x_i für $i < k$ unverändert bleibt, y_i in x_{k+i-1} übergeht, und x_i für $i > k$ in x_{i+m-1} übergeht. Z. B. durch Einsetzung von $\psi(x_2, x_1, x_1, x_3)$ für x_2 in $\phi(x_3, x_2, x_1, x_2, x_4, x_5, x_6)$ hat man $\phi(x_5, \psi(x_3, x_2, x_2, x_4), x_1, \psi(x_3, x_2, x_2, x_4), x_6, x_7, x_8)$. (Wenn man eine Konstante als eine Funktion von 0 Variablen ansieht, so ist der Fall (2) als Spezialfall im Falle (3) eingeschlossen).

Diese Operationen sind die, mit denen wir uns in diesem Kapitel zu beschäftigen haben. Die Gesamtheit der Ausdrücke, die sie erzeugen, hat die Eigenschaft, dass jede Funktion und jede Aussage, die aus den Grundfunktionen und Grundkonstanten gebildet werden kann, mit einem und nur einem Ausdruck bezeichnet wird. Es kann aber geschehen, dass derselbe Ausdruck in mehreren Weisen durch diese Operationen erzielt wird, also dass ganz verschiedene Operationsprozesse in dem Sinne äquivalent sind, dass sie dasselbe Etwas liefern. Diese Äquivalenzen sind die Eigenschaften, die hier in Betracht kommen.

Diese Ausdrücke lassen sich nun in unsere Schreibweise umformen, wenn wir nur die Variablen behandeln, als ob sie Etwase wären. In der Tat geht ein Ausdruck der Form $f(u_1, u_2, \cdots u_n)$ wegen der Ausdeutung der Anwen-

dung (als Schönfinkelscher Funktionsbegriff) in $(fu_1u_2\cdots u_n)$ über. Z. B. wird der oben (in 3) betrachtete Ausdruck nach der Umformung

$$(\phi x_5(\psi x_3x_2x_2x_4)x_1(\psi x_3x_2x_2x_4)x_6x_7x_8).$$

Die neuen Ausdrücke sind also Kombinationen von gewissen Grundfunktionen, Grundkonstanten und Variablen. Diese Kombinationen lassen sich aber ferner aus Kombinationen von lauter Variablen erschaffen und zwar dadurch, dass man in eine der letzten für x_1 eine Grundfunktion nach dem obigen Prozess 2) einsetzt, dann für x_2 eine Grundkonstante oder Grundfunktion einsetzt, u. s. w., bis alle die im betreffenden Ausdruck erscheinenden Grundgegenstände eingesetzt sind. Z. B. wird das oben Geschriebene aus

$$(x_1x_7(x_2x_5x_4x_4x_6)x_3(x_2x_5x_4x_4x_6)x_8x_9x_{10})$$

durch Einsetzung von ϕ statt x_1 und ψ statt x_2 erzielt.

Diese letzte Bemerkung läuft darauf hinaus, dass die Kombinationen lauter Variablen Operatoren sind, die aus den gegebenen Funktionen und Konstanten alle möglichen abgeleiteten Funktionen und Konstanten durch Einsetzung, und zwar die Schönfinkelsche,* hervorrufen. In diesem Kapitel zeige ich, dass diese Operatoren nichts wesentlich anderes als eine bestimmte Klasse von Kombinatoren sind. In der Tat setze ich zuerst fest, dass ein Kombinator Y eine Kombination der Variablen x_1, $x_2\cdots x_n$, nämlich X dann und nur dann *darstellt,* wenn es formal,—d. h. durch Behandlung der Variablen als Etwase ohne besondere Eigenschaften—folgt, das

$$\vdash Yx_1x_2\cdots x_n = X,$$

Dann beweise ich die folgenden Hauptsätze:

I. *Wenn ein Kombinator Y eine Kombination von $x_1x_2\cdots x_n$ darstellt, so stellt er nur eine dar* (bewiesen in C 1).

II. *Zu jeder Kombination lauter Variabler gibt es mindestens einen Kombinator Y, der sie darstellt* (bewiesen in E 1).

III. *Wenn zwei Kombinatoren Y_1 und Y_2 dieselbe Kombination lauter Variabler darstellen, dann folgt es ohne Gebrauch von Variablen, dass* $\vdash Y_1 = Y_2$ (bewiesen in E 3).†

Aus diesen drei Hauptsätzen folgt leicht, was ich beweisen will. Denn zunächst wird jeder Ausdruck eindeutig in der Form $Yu_1u_2\cdots u_n$, wo u_1, $u_2\cdots u_n$ die in dem Ausdruck erscheinenden Grundgegenstände sind,

* Vgl. oben unter I A.

† Mit der unwichtigen Beschränkung dass Y_1 und Y_2 eigentlich (s. unten II E) sind.

dargestellt, und weiter lässt sich jeder Substitutionsprozess so durch Kombinatoren definieren, dass aus den Ausdrücken der Form $Yu_1u_2\cdots u_n$ immer neue Ausdrücke derselben Form erzeugt werden.

Dieses Resultat ist freilich nur ein Spezialergebnis der drei Hauptsätze, welche eine gewisse Art von Isomorphismus zwischen Kombinatoren und Kombinationen überhaupt aussagen. Mit dem Kombinator K kommen Kombinationen mit Auslassungen in Betracht. Wir könnten diese ausschliessen, und nur Kombinatoren, die Kombinationen von B, C, W und I sind, betrachten; aber die allgemeineren Sätze können fast so leicht bewiesen werden als die besonderen, und also habe ich den K behalten. Die Betrachtungen, die den K speziell betreffen, sind unten zuweilen nicht so ausführlich angegeben, wie die anderen.

Eine letzte Bemerkung kommt dazu. In der eben dargestellten Theorie wurden bisher Funktionen von verschiedenen Anzahlen von Variablen unterschieden; dagegen ist jedes Etwas wegen Reg. E eine Funktion beliebig vieler Variablen. Infolgedessen hat man mit jedem Etwas nicht bloss eine bestimmte endliche Anzahl von Leerstellen zu assoziieren, sondern eine unendliche Folge von Leerstellen. Durch irgendeinen der obigen Prozesse wird aber nur eine endliche Anzahl von Leerstellen gestört, also haben diese Folgen einen besonderen Charakter. Dies erklärt die Tatsache, dass von Abschnitt C ab Folgen einer bestimmten Art eine grosse Rolle spielen.

B. Die Grundlegenden Definitionen und Sätze.

§ 1. *Die B Sequenz.*

Def. 1. $B_1 \equiv B$; $\quad B_{n+1} \equiv BBB_n$, $\qquad (n = 1, 2, 3, \cdots)$.

Def. 2. $B_0 \equiv I$.

Satz 1. *Wenn X, Y und Z Etwase sind, dann*

$$\vdash B_{n+1}XYZ = B_nX(YZ), \qquad (n = 0, 1, 2, \cdots).$$

Beweis: Für $n = 0$ folgt der Satz aus Def. 1 und 2, Reg. B, und I D Satz 9.
Es sei $n > 0$;

$$\begin{aligned} \vdash B_{n+1}XYZ &= BBB_nXYZ && \text{(Def. 1, I D Satz 6)},\\ &= B(B_nX)YZ && \text{(Reg. } B\text{)},\\ &= B_nX(YZ), \quad \text{w. z. b. w.} && \text{(Reg. } B\text{)}. \end{aligned}$$

SATZ 2. *Wenn $X, Y, Z_1, Z_2, \cdots, Z_m$ Etwase sind, dann*

$$\vdash B_{n+m}XYZ_1Z_2\cdots Z_m = B_nX(YZ_1Z_2\cdots Z_m), \qquad (m=0,1,2,\cdots;\ n=0,1,2,\cdots).$$

Beweis: Für $m=0$ klar. Für $m=1$ folgt der Satz aus Satz 1.

Angenommen, der Satz sei für $m=k$ bewiesen. Dann wird er für $m=k+1$ bewiesen, wie folgt: aus Satz 1 folgt

$$\vdash B_{n+k+1}XYZ_1\cdots Z_kZ_{k+1} = B_{n+k}X(YZ_1)Z_2Z_3\cdots Z_kZ_{k+1}, \qquad (n=0,1,2\cdots).$$

Nun in den vorliegenden Satz für $m=k$ setzen wir (YZ) für Y, und Z_{i+1} für Z_i $(i=1,2,\cdots,k)$. Dann

$$\vdash B_{n+k}X(YZ_1)Z_2Z_3\cdots Z_kZ_{k+1} = B_nX(YZ_1Z_2\cdots Z_kZ_{k+1}), \qquad (n=0,1,2,\cdots).$$

Aus den letzten zwei Formeln folgt der Satz für $m=k+1$. Durch Wiederholung dieses Prozesses wird der Satz für ein beliebiges m bewiesen.

SATZ 3. *Sind $X, Y, Z_1, Z_2, \cdots, Z_m$ Etwase, dann*

$$\vdash B_mXYZ_1Z_2\cdots Z_m = X(YZ_1Z_2\cdots Z_m), \quad (m=0,1,2,3,\cdots).$$

Beweis: Man setzt $n=0$ in Satz 2.

SATZ 4. *Wenn eine Reihe von Etwasen $X_1, X_2, X_3, \cdots$, einer Rekursionsformel, näm.,*

$$\vdash X_{n+1} = BX_n \qquad (n=1,2,3,\cdots),$$

erfüllen, dann

$$\vdash X_{n+m} = B_mX_n, \qquad (m=0,1,2,\cdots;\ n=1,2,3,\cdots).$$

Beweis: Für $m=0$ oder $m=1$ klar.

Nun sei der Satz für $m=k$ augenommen. Dann folgt

$$\begin{aligned} \vdash X_{n+k+1} &= BX_{n+k} && \text{(Hp.)},\\ &= B(B_kX_n) && \text{(nach diesem Satz für } m=k),\\ &= BBB_kX_n && \text{(Reg. } B),\\ &= B_{k+1}X_n && \text{(Def. 1)}. \end{aligned}$$

Also kann der Satz für ein beliebiges m bewiesen werden.

SATZ 5. *Ist X ein Etwas, dann*

$$\vdash B_m(B_nX) = B_{m+n}X, \qquad (m,n=0,1,2,3,\cdots).$$

Beweis: Für $n = 0$ klar, weil $\vdash B_0X = X$. Sonst setzt man:

$$X_n \equiv B_nX, \qquad (n = 1, 2, 3, \cdots),$$

dann

$$\begin{aligned} \vdash X_{n+1} &= B_{n+1}X = BBB_nX && \text{(Def. 1)}, \\ &= B(B_nX) = BX_n && \text{(Reg. } B\text{)}. \end{aligned}$$

Der Satz folgt dann aus Satz 4.

§ 2. *Die Identitätsfunktion.*

SATZ 1. $\vdash B_mI = I, \qquad (m = 0, 1, 2, \cdots).$

Beweis: In § 1 Satz 4 setzen wir $X_n \equiv I$. Dann

$$\vdash X_{n+1} = BX_n \qquad \text{(Ax. } I_2\text{)}.$$

Daher folgt der Satz aus § 1 Satz 4.

SATZ 2. *Wenn X ein Etwas ist, dann*

$$\vdash B_mIX = X, \qquad (m = 1, 2, 3, \cdots).$$

Beweis:

$$\begin{aligned} \vdash B_mIX &= IX && \text{(Satz 1)}, \\ &= X, && \text{(I D Satz 9)}. \end{aligned}$$

SATZ 3. $\vdash CBI = I.$

Beweis:

$$\begin{aligned} \vdash CBI &= B(BI)I && \text{(Ax. } I_1\text{)}, \\ &= BII = II && \text{(Ax. } I_2\text{)}, \\ &= I. && \text{(I D Satz 9)}. \end{aligned}$$

SATZ 4. *Wenn X ein Etwas ist, dann gilt* $\vdash BXI = X.$

Beweis:

$$\begin{aligned} \vdash BXI &= CBIX && \text{(Reg. } C\text{)}, \\ &= IX && \text{(Satz 3)}, \\ &= X && \text{(I D Satz 9)}. \end{aligned}$$

SATZ 5. *Def.* 1 *von* § 1 *gilt auch, wenn* $n = 0$, *d. h.*

$$\vdash B_1 = BBB_0.$$

Beweis: Klar aus Satz 4 und § 1, Def. 2.

§ 3. *Die C, W und K Sequenzen.*

Def. 1. $C_1 \equiv C; \quad C_{n+1} \equiv BC_n, \qquad (n = 1, 2, 3, \cdots).$

SATZ 1. $\vdash B_mC_n = C_{m+n}, \qquad (m = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots).$

Beweis: Folgt unmittelbar aus Def. 1 und § 1 Satz 4.

SATZ 2. *Wenn* $X_0, X_1, X_2, \cdots, X_n, X_{n+1}$, *Etwase sind, dann*

$$\vdash C_n X_0 X_1 X_2 \cdots X_n X_{n+1} = X_0 X_1 X_2 \cdots X_{n-1} X_{n+1} X_n.$$

Beweis:
$$\begin{aligned} \vdash C_n X_0 X_1 \cdots X_{n-1} X_n X_{n+1} &= B_{n-1} C_1 X_0 X_1 \cdots X_{n+1} && \text{(Satz 1)},\\ &= C_1 (X_0 X_1 X_2 \cdots X_{n-1}) X_n X_{n+1} && \text{(§ 1 Satz 3)},\\ &= X_0 X_1 X_2 \cdots X_{n-1} X_{n+1} X_n, && \text{(Reg. } C\text{)}. \end{aligned}$$

Def. 2. $W_1 \equiv W; \qquad W_{n+1} \equiv B W_n, \qquad (n = 1, 2, 3, \cdots).$

SATZ 3. $\vdash B_m W_n = W_{m+n}, \qquad (m = 0, 1, 2, \cdots; \; n = 1, 2, 3, \cdots).$

Beweis: Folgt unmittelbar aus Def. 2 und § 1, Satz 4.

SATZ 4. *Wenn* $X_0, X_1, X_2, \cdots X_n$ *Etwase sind, so gilt*

$$\vdash W_n X_0 X_1 \cdots X_n = X_0 X_1 \cdots X_{n-1} X_n X_n.$$

Beweis: Wie der von Satz 2, mit Gebrauch von Reg. *W* anstatt Reg. *C*.

Def. 3. $K_1 \equiv K; \qquad K_{n+1} \equiv B K_n, \qquad (n = 1, 2, 3, \cdots).$

SATZ 5. $\vdash B_m K_n = K_{m+n}, \qquad (m = 1, 2, 3, \cdots; \; n = 1, 2, 3, \cdots).$

SATZ 6. *Wenn* $X_0, X_1, X_2, \cdots X_n$ *Etwase sind, so gilt*

$$\vdash K_n X_0 X_1 X_2 \cdots X_n = X_0 X_1 X_2 \cdots X_{n-1}.$$

Beweis: Wie der von Satz 2 mit Gebrauch von Reg. *K* anstatt Reg. *C*.

§ 4. *Das zusammengesetzte Produkt.*

Def. 1. Wenn *X* und *Y* Etwase sind, $(X \cdot Y) \equiv BXY$.

SATZ 1. *Sind X, Y, Z Etwase, dann* $\vdash (X \cdot Y) Z = X(YZ)$.

Beweis: Folgt aus Def. 1 und Reg. *B*.

Def. 2. Wenn *X* und *Y* Bezeichnungen für Etwase sind, und keine von den beiden von der Form $(U = V)$ oder $(U \cdot V)$ ist, so dürfen wir in einer Bezeichnung wie $X \cdot Y$ die äusseren Klammern von *X* und *Y* fortlassen.

SATZ 2. *Sind X und Y Etwase, dann gilt*

$$\vdash B(X \cdot Y) = BX \cdot BY.$$

Beweis:

(1) $\vdash B(X\cdot Y)=B(BXY)$ (Def. 1),
$=B_2BBXY$ (§ 1, Satz 3).

(2) $\vdash BX\cdot BY=B(BX)(BY)$ (Def. 1 und 2),
$=B_2X(BY)$ (§ 1, Satz 5, $m=n=1$),
$=B_3XBY$ (§ 1, Satz 2, $m=1$, $n=2$),
$=CB_3BXY$ (Reg. C).

Nun aus § 1, Def. 1,

$$\vdash CB_3B=C(BB(BBB))B,$$

daher aus Ax. B

(3) $\vdash CB_3BXY=B(BB)BXY$
$=B_2BBXY$ (§ 1, Satz 5).

Aus (1), (2), (3), wird der Satz bewiesen.

SATZ 3. *Das Produkt* $(X\cdot Y)$ *ist assoziativ, d. h., wenn* X, Y, Z *Etwase sind, so gilt*

$$\vdash X\cdot(Y\cdot Z)=(X\cdot Y)\cdot Z.$$

Beweis:

$\vdash X\cdot(Y\cdot Z)=B(X\cdot Y)Z$ (Def. 1),
$=(BX\cdot BY)Z$ (Satz 2),
$=BX(BYZ)$ (Satz 1),
$=X\cdot(Y\cdot Z)$ (Def. 1).

SATZ 4. *Wenn* X *ein Etwas ist, so gelten*

1) $\vdash I\cdot X=X$,
2) $\vdash X\cdot I=X$.

Beweis: Folgt aus Def. 1 und § 2, Sätze 2 und 4.

SATZ 5. $\vdash B_m\cdot B_n=B_{m+n}$, $(m, n=1, 2, 3, \cdots)$.

Beweis: Für $m=1$ folgt aus Def. 1 und § 1, Def. 1. Ist nun der Satz für $m=k$ bewiesen, dann folgt

$\vdash B_{k+1}\cdot B_n=(B\cdot B_k)\cdot B_n$ (nach diesem Satze für $m=1$),
$=B\cdot(B_k\cdot B_n)$ (Satz 3),
$=B\cdot B_{n+k}$ (nach diesem Satze für $m=k$),
$=B_{n+k+1}$, (nach diesem Satze für $m=1$).

Dadurch wird der Satz für ein beliebiges m bewiesen.

SATZ 6. *Sind X und Y Etwase, dann gilt,*

$$\vdash B_m(X \cdot Y) = B_m X \cdot B_m Y, \qquad (m = 1, 2, 3, \cdots).$$

Beweis: Für $m = 1$ ist der Satz mit Satz 2 identisch. Nun sei der Satz für $m = k$ angenommen, dann wird er für $m = k + 1$ bewiesen, wie folgt:

$$\begin{aligned} \vdash B_{k+1}(X \cdot Y) &= B(B_k(X \cdot Y)) && (\S\,1, \text{ Satz } 5), \\ &= B(B_k X \cdot B_k Y) && (\text{nach diesem Satze für } m = k), \\ &= B(B_k X) \cdot B(B_k Y) && (\text{Satz } 2), \\ &= B_{k+1} X \cdot B_{k+1} Y, && (\S\,1, \text{ Satz } 5). \end{aligned}$$

SATZ 7. $\vdash B_m B_n \cdot B_m B_p = B_m B_{n+p}, \quad (m, n, n, = 0, 1, 2, \cdots).$

Beweis: Folgt aus Sätzen 4,* 5, 6.

Def. 3. Sind $X_1, X_2, \cdots, X_n$ Etwase, dann

$$(X_1 \cdot X_2 \cdot X_3 \cdots X_n) \equiv (\cdots(((X_1 \cdot X_2) \cdot X_3) \cdot X_4) \cdots X_n).$$

Festsetzung: Da ich die Assoziativität des Produkts $(X \cdot Y)$ bewiesen habe, brauche ich diese Tatsache nicht immer explizit hervortreten zu lassen. Im Gegenteil werde ich die Form der Def. 3 benutzen, und dann habe ich ein Recht, Klammern nach Belieben einzusetzen. Die Def. 3 setzt aber an sich die Assoziativität nicht voraus, und wird benutzt, wo die Assoziativität nicht behauptet ist (z. B. s. die nächste Nummer).

§ 5. *Die Axiome in der neuen Darstellungsweise.*

SATZ. *Wenn man die Definitionen von §§ 1-4 berücksichtigt, nehmen die kombinatorischen Axiome die folgende Gestalt an:*

a. *Kommutative Axiome.*

Ax. B. $\vdash CB_3B = BB \cdot B.$
Ax. C. $\vdash CB_3C = BC \cdot B_2.$
Ax. W. $\vdash CB_2W = BW \cdot B_2.$
Ax. K. $\vdash CB_2K = BK \cdot I.$
Ax. I_1. $\vdash CBI = BI \cdot I.$

b. *Transmutative Axiome.*

Ax. (BC). $\vdash B_1 \cdot C_1 = C_2 \cdot C_1 \cdot BB.$
Ax. (BW). $\vdash B_1 \cdot W_1 = W_2 \cdot W_1 \cdot C_2 \cdot B_2 B \cdot B.$
Ax. (BK). $\vdash B_1 \cdot K_1 = K_1 \cdot K_1.$

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* Satz 4 ist nur wenn $n = 0$ oder $p = 0$ notwendig.

Ax. $(CC)_1$. $\vdash C_1 \cdot C_1 = B_2 I$.
Ax. $(CC)_2$. $\vdash C_2 \cdot C_1 \cdot C_2 = C_1 \cdot C_2 \cdot C_1$.
Ax. (CW). $\vdash C_1 \cdot W_1 = W_2 \cdot C_1 \cdot C_2$.
Ax. (CK). $\vdash C_1 \cdot K_1 = K_2$.
Ax. (WC). $\vdash W_1 \cdot C_1 = W_1$.
Ax. (WW). $\vdash W_1 \cdot W_1 = W_1 \cdot W_2$.
Ax. (WK). $\vdash W_1 \cdot K_1 = BI$.

c. *Andere Axiome.*

Ax. I_2. $\vdash BI = I$.

Bemerkungen über die Bedeutung dieser Axiomen. Diese Axiomen haben einen Vertauschbarkeitscharakter, d. h., sie sagen aus, dass gewisse Ausdrücke in zwei Weisen gebildet werden können. Z. B. kann der Ausdruck

$$x_0 x_3 (x_1 x_2)$$

aus dem Ausdruck

$$x_0 x_1 x_2 x_3$$

in den folgenden zwei Weisen gebildet werden: 1), man setzt zunächst die Klammern ein, und führt dann eine Vertauschung aus, oder 2), man macht zuerst eine Vertauschung und dann eine Einsetzung von Klammern. Diese zwei Konstructionsweisen entsprechen den zwei Darstellungen

$$(B \cdot C) x_0 x_1 x_2 x_3$$
$$(C_2 \cdot C_1 \cdot BB) x_0 x_1 x_2 x_3$$

die nach Ax. (BC) gleich sind. Ebenso sagen die sämtlichen transmutativen Axiomen eine Vertauschbarkeit aus. Dass die kommutativen Axiomen in dieser Hinsicht nicht wesentlich verschieden sind, folgt aus II D 2, Satz 1 (s. unten).

Der Unterschied zwischen den kommutativen und transmutativen Axiomen liegt darin, dass die Kombinatoren, deren Vertauschbarkeit durch jene Axiomen ausgesagt wird, nicht miteinander übergreifen. Um dies zu erklären, setzen wir voraus, dass für einen Kombinator X gilt

$$\vdash X x_0 x_1 x_2 \cdots x_m = x_0 y_1 y_2 \cdots y_n \text{ *}$$

wo $y_1, y_2, \cdots, y_n$ Kombinationen von $x_1, x_2, \cdots, x_m$ sind. Ist Y irgendein

* Diese Gleichung ist in dem Sinne zu verstehen, dass die Variablen wie beliebige Etwase betrachtet werden.

Kombinator der Form B_mZ, so ist es ersichtlich, dass Y auf Variablen wirkt, die durch X nicht gestört werden, und umgekehrt; also muss eine Gleichung der Form

$$B_mZ \cdot X = X \cdot B_nZ$$

inhaltlich richtig sein. Diese Gleichung sagt aber eine Vertauschbarkeit derjenigen Art aus, welche aus den kommutativen Axiomen folgt. Die transmutativen Axiomen haben aber den entgegengesetzten Charakter. Z. B. wirken die zwei Kombinatoren B und C, die auf der linken Seite des $Ax.$ (BC) erscheinen, auf dieselben Variablen, näm. x_1 und x_2.

Das Ax I_2 ist ein Axiom ganz anderer Art. Seine wesentliche Bedeutung ist dass es uns ermöglicht, die Gleichheit von Kombinatoren zu beweisen, die im Sinne von II C 1 (unten) derselben Folge von lauter Variablen, aber mit verschiedenen Ordnungen entsprechen—was sonst unmöglich wäre (II C 1, Satz 6). D. h., dieses Axiom sagt aus, dass ein Kombinator unabhängig davon zu verstehen ist, wie viele Variablen hinzuzufügen sind, damit eine Reduktion auf eine Kombination von lauter Variablen sich vollzieht.

A Property of Unbounded Continua, with Applications.*

By W. A. Wilson.

1. A great part of the theory of continua is valid only if the continua under discussion are bounded. This is primarily due, of course, to the fact that for unbounded sets it is not necessarily true that every infinite set has at least one limiting point. Hence such fundamental theorems as the Cantor divisor theorem and the existence of various kinds of irreducible continua fail.† However, many theorems which are true only for bounded continua are readily generalized for the unbounded case by the imposition of suitable restrictions. The favorite tool used for such extensions has been the method of inversion.‡

It is the purpose of this article to call attention to a property of unbounded continua (§ 2) which is useful in investigations of this kind and to give applications of it to irreducible continua and the separation of the plane. It is possible that this property has been noted before by other students of the subject (Cf. the article by Knaster and Kuratowski referred to above and the article by Szymanski referred to in § 5), but, to judge by the literature, it has been discarded as of no importance. The material of §§ 2-5 is valid in any metric space for which the Bolzano-Weierstrass property that every bounded infinite set has at least one limiting point holds; for the sake of brevity such a space will be called a W-space.

2. Theorem. *Let $\{X_i\}$ be a sequence of continua lying in a W-space and let the upper closed limit K of the sequence be a non-void. Let each X_i be unbounded or let the diameter of X_i increase indefinitely with i. Then K is an unbounded continuum or a closed set each component of which is an unbounded continuum.*

* Presented to the American Mathematical Society, December, 1929.

† A conspicuous exception is that in a metric connected space, which is everywhere locally connected and in which every bounded infinite set has one or more limiting points, every closed cut between two points contains an irreducible cut between the points.

‡ The principal properties of inversion may be found systematically worked out in the following articles: C. Kuratowski, "Sur la méthode d'inversion dans l'Analysis Situs," *Fundamenta Mathematicae*, Vol. 4, pp. 151-163; and B. Knaster and C. Kuratowski, "Sur les continus non-bornés," *Ibid.*, Vol. 5, pp. 23-58.

Proof. Let us assume the opposite, namely that K has a bounded component A. Let $\epsilon > 0$, S denote the set of points whose distances from A are less than ϵ, and F be the frontier of S. By the definition of upper closed limit there is for each point a of A a partial sequence $\{i'\}$ of $\{i\}$ and a set of points $\{a_{i'}\}$ such that each $a_{i'}$ lies in $X_{i'}$ and $a_{i'} \to a$. For i' large enough every $a_{i'}$ lies in S and $X_{i'}$ contains points not in $\bar{S}$. Since $\bar{S}$ is bounded, $X_{i'}$ contains a sub-continuum $Y_{i'}$ contained in $\bar{S}$ and joining $a_{i'}$ to a point $y_{i'}$ of F. Since the union of the continua $\{Y_{i'}\}$ is bounded, the upper closed limit of the sequence $\{Y_{i'}\}$ is non-void and is a continuum joining a to a point y on F. But, as $Y \subset K$, it follows that $Y \subset A$. This is a contradiction, for $A \subset S$ and y lies in F.

COROLLARY. *Let $\{X_i\}$ be a monotone decreasing sequence of unbounded continua lying in a W-space and let D be their divisor. If D is not void, it is an unbounded continuum or a closed set of which each component is an unbounded continuum.*

Note 1. If in the above theorem we impose no restrictions on the diameters of the continua $\{X_i\}$, but require that the lower closed limit of the sequence be non-void, we find that K is either a continuum or a closed set of which each component is an unbounded continuum. The proof is a modification of that given above.

Note 2. A consequence of the above corollary is that in C. Kuratowski's decomposition of the irreducible continuum into *tranches,** each tranche is a continuum or a closed set of which each component is an unbounded continuum.

3. A generalization. It is convenient to say that an unbounded continuum containing points of a set α joins α to ∞. If a continuum K joins a set α to ∞ and no proper sub-continuum has this property, K is called *irreducible between α and ∞*. If a continuum is irreducible between two sets α and β, and also between α and ∞, we say that it is irreducible between α and $\beta + \infty$. The following examples, all of which are sets in the Cartesian plane, make this notion clearer.

Example I. Let H be the set defined by $0 < x \leqq 1$ and $y = 1/x$. Then H is a continuum irreducible between $(1, 1)$ and ∞.

Example II. Let K be the set defined by $0 < x \leqq 1$ and $y = 1/x \sin^2(\pi/x)$.

* "Théorie des continus irréductibles entre deux points II," *Fundamenta Mathematicae*, Vol. 10, p. 254.

Then K is not a continuum, although it is homeomorphic with the set H in Ex. I.

Example III. Let $P = \bar{K}$, where K is the set in Ex. II. Let $\alpha = (1, 0)$ and β be any set for which $x = 0$ and $y \geqq 0$. Then P is a continuum irreducible between α and $\beta + \infty$.

Example IV. Let P be as in Ex. III and Q be the set symmetrical to P with respect to the origin. Set $M = P + Q$, $\alpha = (1, 0)$ and $\beta = (-1, 0)$. Then M is a continuum irreducible between α and β, but not between α or β and ∞.

Example V. Let M be defined thus: for $x = 0$ and $x = 1$, $y \geqq 0$; for $x = 1/n$ and $x = 1 - 1/n$, $0 \leqq y \leqq n$; for $1/n < x < 1 - 1/n$, $y = n$; for $1/(2n+1) < x < 1/2n$ and $1 - 1/(2n+1) < x < 1 - 1/(2n+2)$, $y = 0$. (Here $n = 1, 2, \cdots$) Then the continuum M contains no continuum irreducible between a set α on the line $x = 0$ and a set β on the line $x = 1$; but, if γ is any bounded closed sub-set which has no points on the lines $x = 0$ and $x = 1$, M contains a sub-continuum irreducible between γ and $\alpha + \beta + \infty$.

It is a simple matter to show that, if K is a continuum irreducible between a point a and ∞, and is everywhere locally connected, then K is the homeomorphic image of a ray. Furthermore, if K is simultaneously irreducible between the sets α and β, between α and ∞, and between β and ∞, it is indecomposable.

4. THEOREM. *Let the unbounded continuum M lie in a W-space and contain points of the bounded closed set α. Then M contains a continuum irreducible between α and ∞.*

Proof. Brouwer's theorem for establishing the existence of closed sets irreducible with respect to a property is valid here. For, if $\{K_i\}$ denotes any descending sequence of sub-sets of M, each of which contains points of α and has only unbounded components, and D is the divisor of this sequence, we have these results: $D \cdot \alpha \neq 0$, since each set $\alpha \cdot K_i$ is a bounded closed set; D need not be a continuum, but the components of D are all unbounded continua by § 2 and at least one of them contains points of α.

Note. This theorem need not be valid if α is unbounded or not closed. For example, let M be the positive half of the x-axis and $\alpha = \{n\}$, where $n = 1, 2, \cdots$.

5. THEOREM. *Let M be an unbounded continuum in a W-space which contains points of the bounded closed sets α and β. Let M have a sub-*

continuum C which contains points of α and β, but does not contain two sub-continua without common points which join α and β, respectively, to ∞. Then there is a sub-continuum of M irreducible between α and β.

Proof. The method of § 4 applies. For, if $\{K_i\}$ is a descending sequence of sub-continua of C which contain points of α and β, and D is the divisor of this sequence, D contains a continuum joining α to β, as otherwise, by § 2, it would have at least two unbounded components, one containing points of α, and the other points of β.

Note. The condition given in the above theorem is obviously not necessary. It is closely related to a lemma proved by P. Szymanski † by means of inversion, which may be stated thus: Let A and B be closed sets and A be bounded; let S be a closed set such that $S \cdot A \neq 0 \neq S \cdot B$ and there is no decomposition of S into two closed sets M and N such that $M \cdot N = M \cdot B = N \cdot A = 0$. If S contains no sub-continuum joining A to B, then S contains an unbounded continuum K such that $K \cdot A \neq 0$.

6. Some properties of inversion. If A is a set in a Euclidean space and v is a center of inversion not on A, the inverse of A will be denoted by A^*; whence $A^{**} = A$. This notation and the properties of inversion obtained in the articles of Knaster and Kuratowski referred to in § 1 will be freely used in the following sections.

The following theorems regarding irreducible continua in Euclidean spaces are easily demonstrated:

I. *Let α be a bounded closed set, M a continuum irreducible between α and ∞, and v a point not on $M + \alpha$. Then $M^* + v$ is a continuum irreducible between α^* and v.*

II. *Let the bounded continuum M be irreducible between the closed sub-sets α and β, and v be a point of β. Then $(M - v)^*$ is an unbounded continuum irreducible between α^* and $(\beta - v)^* + \infty$.*

III. *Let the bounded continuum M be irreducible between the closed sub-sets α and β, and v be a point of $M - (\alpha + \beta)$. If $M - v$ is connected, $(M - v)^*$ is an unbounded continuum irreducible between α^* and β^*. If $M - v$ is not connected, $(M - v)^*$ is the sum of two continua P^* and Q^* without common points, which are irreducible between α^* and β^*, respectively, and ∞.*

These theorems lead at once to the following definitions. Let the set M

† "La somme de deux continus irréductibles," *Fundamenta Mathematicae*, Vol. 11, p. 7.

have points in common with each of the sets α and β. Then M is called *unstable with respect to α and β,* if it contains two unbounded continua P and Q such that $P \cdot Q = 0$, $P \cdot \alpha \neq 0$, and $Q \cdot B \neq 0$. Otherwise M is *stable* with respect to the two sets. This distinction is of especial importance in the inversion of irreducible continua, as is shown by the following theorem, which is a modification of one by P. Szymanski (*loc. cit.*, p. 7).

IV. *Let the unbounded continuum M be irreducible between the closed sub-sets α and β. Let v be a point not on M. For $M^* + v$ to be irreducible between α^* and β^* it is necessary and sufficient that M be stable with respect to α and β.*

7. Separation of the plane. Since the property of being an irreducible cut of the plane is an invariant under inversion, the theorems of the previous section enable one to extend to unbounded continua certain theorems regarding frontier sets in the plane.*

THEOREM. *Let $n \geqq 2$ be an integer and let the unbounded continuum F be the union of two continua H and K having one of the following sets of properties:* (1) *$H \cdot K$ is the sum of $n-1$ bounded closed sets $\{\alpha_i\}$ and H and K are irreducible between each pair of the sets $\{\alpha_i\}$ and between each α_i and ∞;* (2) *$H \cdot K$ is the sum of n bounded closed sets $\{\alpha_i\}$, H and K are irreducible between each pair of these, H is bounded, and K is stable with respect to each pair; or* (3) *$H \cdot K$ is the sum of $n-1$ bounded closed sets $\{\alpha_i\}$ and another closed set β, and both H and K are irreducible between each pair of sets $\{\alpha_i\}$ and between each α_i and $\beta + \infty$. Then F is the frontier of at least n components of its complement.*

Proof. Invert the plane with respect to a point v not on F. In Case 1, H and K are stable with respect to each pair of closed sets $\{\alpha_i\}$; hence by § 6, Theorems I and IV, $H^* + v$ and $K^* + v$ are irreducible between each pair of the n closed sets v and $\{\alpha_i^*\}$, while $(H^* + v) \cdot (K^* + v) = v + \sum_1^{n-1} \alpha_i^*$. In Case 2 for like reasons H^* and $K^* + v$ are irreducible between each pair of the n closed sets $\{\alpha_i^*\}$; and $H^* \cdot (K^* + v) = \sum_1^n \alpha_i^*$. In Case 3 both H and K are stable with respect to each pair of the n sets β and $\{\alpha_i\}$. Hence $H^* + v$ and $K^* + v$ are irreducible between each pair of the n sets $\{\alpha_i^*\}$ and $\beta^* + v$, while $(H^* + v) \cdot (K^* + v) = H^* \cdot K^* + v = \beta + v + \sum_1^{n-1} \alpha_i^*$.

Thus in every case $F^* + v$ is the union of two bounded continua whose divisor is the sum of n closed sets between each pair of which both continua are irreducible. Hence $F^* + v$ is the frontier of at least n components of its

* W. A. Wilson, "On Bounded Regular Frontiers," *Bulletin of the American Mathematical Society*, Vol. 34, p. 86, and C. Kuratowski, "Sur la séparation d'ensembles situés sur le plan," *Fundamenta Mathematicae*, Vol. 12, p. 235.

complement by the theorem cited in the foot-note above, and by inversion this is also true of F.

THEOREM. *Conversely, let F be a decomposable unbounded plane continuum which is the frontier of n ($\geqq 2$) components of its complement. Then F satisfies one of the sets of conditions of the previous theorem.*

Proof. Invert with respect to a point v not on F. Then $F^* + v$ is a decomposable bounded regular frontier which is the union of two continua H' and K' such that $H' \cdot K'$ is the sum of n closed sets $\{\alpha_i'\}$ between each pair of which both H' and K' are irreducible. There are four cases.

I. Some α_i', say α_n', is the point v. Invert with respect to v again, setting $H = (H' - v)^*$ and $K = (K' - v)^*$. Then $H \cdot K = \sum_1^{n-1} \alpha_i$, where $\alpha_i = \alpha_i'^*$ and each α_i is bounded. By § 6, Theorem II, both H and K are irreducible between each pair of the sets $\{\alpha_i\}$, $i \leqq n - 1$, and between each α_i and ∞. Thus we have Case 1 of the previous theorem.

II. v is a point of some α_i', say of α_n', and $\alpha_n' - v \neq 0$. Proceed as in the first case, setting $\alpha_i = \alpha_i'^*$ for $i < n$, and $\alpha_n = (\alpha_n' - v)^*$. Since α_n' is closed, so is α_n. Here $H \cdot K = \sum_1^n \alpha_i$, and H and K are irreducible between each pair of the sets $\{\alpha_i\}$ for $i < n$, and between each α_i for $i < n$ and $\alpha_n + \infty$. Thus we have Case 3 of the previous theorem.

III. The point v lies in $H' - H' \cdot K'$ or in $K' - H' \cdot K'$, say the latter, and does not disconnect K'. Invert again, setting $H = H'^*$ and $K = (K' - v)^*$. Then each $\alpha_i = \alpha_i'^*$ is a bounded set and both H and K are irreducible between each pair of the sets $\{\alpha_i\}$. As $H' \cdot v = 0$, H is bounded. As $K' = K^* + v$, K is stable with respect to each pair of the sets $\{\alpha_i\}$ by § 6, Theorem IV. Thus we have Case 2 of the previous theorem.

IV. The point v lies in $K' - H' \cdot K'$, as above, and disconnects K'. Then K' is decomposable, and this can only happen when $n = 2$. This gives $K' = P' + Q'$, where $P' \cdot Q' = v$ and P' and Q' are continua irreducible between α_1' and α_2', respectively, and v. Then $F^* + v = P' + (H' + Q')$, $P' \cdot (H' + Q') = \alpha_1' + v$, and the continua P' and $H' + Q'$ are irreducible between α_1' and v. Invert again, setting $P = P'^*$, $H + Q = (H' + Q')^*$, and $\alpha_1 = \alpha_1'^*$. Then $P \cdot (H + Q) = \alpha_1$ and both P and $H + Q$ are continua irreducible between α_1 and ∞ by § 6, Theorem II. Thus we have Case 1 of the previous theorem once more.

Remarks. At first sight the three sets of conditions satisfied by the frontier in the above theorems appear to be unnecessarily complicated. But examples can be constructed showing the effectiveness of each set of conditions; it does not seem worth while, however, to give a full set of these, since for the case that $n = 2$ we can find examples for all three sets by merely modifying the familiar graph of $y = \sin 1/x$.

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Centers of Symmetry in Analysis Situs.*

By Harry Merrill Gehman.

1. *Introduction.* The results of this paper were suggested by the fact that a simple continuous arc is "symmetrical" with respect to any one of its interior points in the sense that the complement of an interior point consists of two mutually separated sets which are homeomorphic † in such a way that the homeomorphism between them can be extended to the center of symmetry.

A new characterization of a simple continuous arc is given in Part 5, using the type of center of symmetry which I have called a "symmetrical cut point."

2. *Symmetry in a Euclidean n-Space.* Thruout the present section, let M denote a point set lying in an n-dimensional Euclidean space S. We shall say that P is a *center of symmetry of* M (in the sense of Analysis Situs), if there exists in S an n-dimensional rectangular coördinate system T having P as origin, which has the property that if $A=(x_1, x_2, \cdots, x_n)$ is a point of M, then $A_T=(-x_1, -x_2, \cdots, -x_n)$ is also a point of M. It will be convenient to refer to A and A_T as a *pair of corresponding points.* If P is itself a point of M, the points P and P_T are identical, but in all other cases, the points of a pair of corresponding points are distinct.

From this definition of center of symmetry, a number of obvious theorems can be deduced, of which the two following are of some interest.

Theorem 1. *If P is a center of symmetry of the point set M, and if P is neither a point nor a limit point of M, then every point within an n-dimensional sphere enclosing P but enclosing no point of M, is also a center of symmetry of M.*

This is true because a new coördinate system T' can be set up, which effects a change of origin from P to any other given point within the sphere, but does not affect the coördinates of points on and outside the sphere.

Theorem 2. *If P is a center of symmetry of M and if $(M+P)-P$*

* Presented to the American Mathematical Society at Ann Arbor, Michigan, on November 29, 1929.

† The sets M_1 and M_2 are said to be *homeomorphic* if there exists a continuous $(1-1)$ correspondence Π such that $\Pi(M_1)=M_2$.

is expressed in any way as the sum of two sets M_1 and M_2 which are such that neither set contains both points of any pair of corresponding points, then there exists a continuous $(1-1)$ *correspondence* Π *having these properties:* $\Pi(S)=S$, $\Pi(M)=M$, $\Pi(M_1)=M_2$, $\Pi(P)=P$, and $\Pi^{-1}(A)=\Pi(A)$, *for each point A of S.*

The proof of this theorem is obvious, since the correspondence may be defined as a reflection of space about P as center, with respect to the coördinate system T.

3. *Symmetry in a Point Set.* Thruout the remainder of this paper, we shall let M denote a point set which comprises the totality of points under consideration. That is, M is now thought of as a "space," and no properties are assumed of another space within which M may lie. Center of symmetry cannot now be defined thru the intermediation of a coördinate system, and we shall use instead certain properties of continuous $(1-1)$ correspondences as a basis for our definition of center of symmetry.

Definition. We shall say that the point P is a *center of symmetry of M,* if $M-P$ can be expressed as the sum of two mutually exclusive sets M_1 and M_2 which are such that there exists a continuous $(1-1)$ correspondence Π having the properties: $\Pi(M)=M$, $\Pi(M_1)=M_2$, $\Pi(P)=P$, and $\Pi^{-1}(A)=\Pi(A)$, for each point A of M.

It might be interesting to point out in connection with this definition that if PQ is a simple continuous arc, and M denotes the set $PQ-Q$, then $M-P$ may be expressed as the sum of two sets M_1 and M_2 and a continuous $(1-1)$ correspondence Π may be defined in such a way that all the properties of this correspondence that are necessary to make P fulfill the definition of center of symmetry of M, are fulfilled, excepting the last. This is illustrated by the following example.

Example 1. On the x-axis, let M denote the set of points $0 \leqq x < 1$. Let P be the origin. Let P_i $(i=1, 2, \cdots)$ be the point $1/2^i$, and let Q_i $(i=1, 2, \cdots)$ be the point $1-(1/2^{i+1})$. Let M_1 consist of the points P_{2i} and Q_{2i-1} and the segments Q_1P_1, $P_{2i}P_{2i+1}$, $Q_{2i+1}Q_{2i}$, for $i=1, 2, \cdots$. Finally let $M_2=M-P-M_1$. We shall define the continuous $(1-1)$ correspondence Π as follows: If $0 \leqq x \leqq 1/2$, then $\Pi(x)=x/2$; if $1/2 \leqq x \leqq 3/4$, then $\Pi(x)=x-(1/4)$; if $3/4 \leqq x < 1$, then $\Pi(x)=2x-1$. It is easily seen that $\Pi(M)=M$, $\Pi(M_1)=M_2$, $\Pi(P)=P$, but $\Pi^{-1}(A) \neq \Pi(A)$ for each point of $M-P$.

4. *Symmetrical Cut Points.* We shall say that the point P is a *symmetrical cut point of* M, if $M-P$ can be expressed as the sum of two mutually separated sets M_1 and M_2, which are such that there exists a continuous $(1-1)$ correspondence Φ having these properties: $\Phi(M_1+P)=M_2+P$ and $\Phi(P)=P$.

THEOREM 3. *If P is a symmetrical cut point of M, then P is a center of symmetry of M.*

Proof. If P is a symmetrical cut point of M, a continuous $(1-1)$ correspondence Π can be defined thus: for each point A of M_1+P, $\Pi(A)=\Phi(A)$; for each point A of M_2, $\Pi(A)=\Phi^{-1}(A)$. The correspondence Π as defined thus has the properties that are necessary in order for P to be a center of symmetry of M.

THEOREM 4. *If M is a point set containing two points P and Q such that* (1) *$M-P$ is not connected and can be expressed as the sum of two mutually separated sets M_1 and M_2 which are homeomorphic, and* (2) *$M-Q$ is connected, then M is connected.*

Proof. Since $M-P$ is not connected and $M-Q$ is connected, the points P and Q are distinct. Hence the set $M-P=M_1+M_2$ contains Q, and is accordingly not vacuous. But since $M_1M_2=0$ and M_1 and M_2 are homeomorphic, it follows that $M-P$ contains at least two points. Hence $M-(P+Q)$ is not vacuous.

Suppose M_1 is the set containing Q, and let R be the point of M_2 which corresponds to Q under the correspondence between M_1 and M_2. If Q is not a limit point of $M-Q$, it is not a limit point of M_1-Q, and hence R is not a limit point of M_2-R. But in that case, the set $M-Q$ can be expressed as the sum of the two mutually separated sets $(M_1-Q)+P+(M_2-R)$ and R. But this is impossible, since by hypothesis $M-Q$ is connected.

Therefore Q is a limit point of the connected set $M-Q$, and the set M is connected.

COROLLARY 4A. *If P is a symmetrical cut point of M, and M contains a point Q such that $M-Q$ is connected, then M is connected.*

THEOREM 5. *If the point set M contains two distinct points A and B such that* (1) *$M-A-B\neq 0$,* (2) *every point of $M-A-B$ is a symmetrical cut point of M, and* (3) *$M-A$ is connected, then M is irreducibly connected between A and B.*

Proof. Let P be a point of $M - A - B$. Then by (2), P is a symmetrical cut point of M, and by (3) and Corollary 4A, M is connected.

Let M_1 be that one of the subsets of $M - P$ which contains A. If the set $M_1 + P - A$ were disconnected, it could be expressed as the sum of the two mutually separated sets H_1 and H_2, where H_2 contains P. But in that case $M - A = (M_1 + P + M_2) - A = (M_1 + P - A) + M_2$ could be expressed as the sum of the two mutually separated sets H_1 and $H_2 + M_2$, contrary to (3). Hence $M_1 + P - A$ is connected.

Hence $\Phi(M_1 + P - A) = M_2 + P - \Phi(A)$ is connected. Since $\Phi(A) \neq P$, it follows that $M - \Phi(A)$ is the sum of the two connected sets $M_1 + P$ and $M_2 + P - \Phi(A)$ having P in common, and $M - \Phi(A)$ is therefore connected. It follows from (2) that $\Phi(A) = B$. Hence $M - B$ is connected and B is a point of M_2.

In other words, each point P of $M - A - B$ separates A and B in M, and hence M is irreducibly connected between A and B.

5. *A New Characterization of a Simple Continuous Arc.* If M is a simple continuous arc from A to B, the hypotheses of Theorem 5 are obviously satisfied. The following example shows however, that a set may satisfy these hypotheses, without necessarily being an arc. We give in Theorem 6, certain additional conditions which are sufficient to characterize an arc.

Example 2. In a plane, let $A = (1, 0)$, $B = (-1, 0)$, $P_n = (n/n + 1, 0)$, $Q_n = (-n/n + 1, 0)$. Let E_n be the curve $y = \sin 1/(nx + x - n)(nx + 2x - n - 1)$, for $n/n. + 1 < x < n + 1/n + 2$, and let F_n be the curve $y = \sin 1/(nx + x + n)(nx + 2x + n + 1)$ for $-(n + 1)/n + 2 < x < -n/n + 1$. Let $M = A + B + \sum_{i=0}^{i=\infty} (P_n + Q_n + E_n + F_n)$. It is easily seen that M is irreducibly connected between A and B, and that every point of $M - A - B$ is a symmetrical cut point of M.

THEOREM 6. *These three conditions are logically equivalent:*

(a) *M is a simple continuous arc from A to B;*

(b) *M is a closed point set containing two points A and B such that $M - A - B \neq 0$, every point of $M - A - B$ is a symmetrical cut point of M, and $M - A$ is connected;*

(c) *M is a regular (connected im kleinen) point set containing two points A and B such that $M - A - B \neq 0$, every point of $M - A - B$ is a symmetrical cut point of M, and $M - A$ is connected.*

The equivalence of conditions (a) and (b) follows from Theorem 5 and theorems due to Lennes and Hallett.* The equivalence of (a) and (c) follows from Theorem 5 and a theorem due to G. T. Whyburn.†

In comparing characterizations (b) and (c) with the previously known characterizations of an arc, it will be noted that boundedness is not assumed, nor is connectivity explicitly assumed. In many respects it resembles the characterization of Sierpinski,‡ with the homeomorphism of $M_1 + P$ and $M_2 + P$ taking the place of Sierpinski's condition of boundedness.

6. *Symmetrical Cut Points of an Unbounded Continuum.* Every point of an open curve M is a symmetrical cut point of M, but if it is known that every point of an unbounded continuum M is a symmetrical cut point of M, it does not follow that M is an open curve, as the following example will show.

Example 3.§ In a plane let K denote the positive half of the x-axis including $x = 0$. Let K_{10j} $(j = 0, \pm 1)$ denote the point set consisting of the segment from $(1, 0)$ to $(2, j)$ and all points of the line $y = j$ for $x \geqq 2$. Let K_{2ij} $(i = 0, \pm 1; j = 0, \pm 1)$ denote the set consisting of the segment from $(3, i)$ to $[4, (3i + j)/3]$ and all points of the line $y = (3i + j)/3$ for $x \geqq 4$. Continue this process. In general, let K_{nij} $[i = 0, \pm 1, \pm 2, \cdots, \pm(3^{n-1} - 1)/2; j = 0, \pm 1]$ denote the set consisting of the segment from $(2n - 1, i/3^{n-2})$ to $[2n, (3i + j)/3^{n-1}]$ and all points of the line $y = (3i + j)/3^{n-1}$ for $x \geqq 2n$. Let $N = K + \sum_{n=1}^{n=\infty} \sum_{i=-R_n}^{i=+R_n} \sum_{j=-1}^{j=+1} K_{nij}$, where $R_n = (3^{n-1} - 1)/2$, and let N_i denote the set obtained by rotating N in the plane about the origin as center thru an angle of $\pi i/2$.

If $M = N_0 + N_1 + N_2 + N_3$, then M is an unbounded continuum, every point of which is a symmetrical cut point of M.

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* N. J. Lennes, *American Journal of Mathematics*, Vol. 33 (1911), p. 308; G. H. Hallett, Jr., *Bulletin of the American Mathematical Society*, Vol. 25 (1919), p. 325.

† *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 688.

‡ *Annali di Matematica*, Ser. 3, Vol. 26 (1916), pp. 131-150.

§ The example is due to Professor J. R. Kline. In describing it, I have made use of the description of a more general set by W. L. Ayrès. See his paper in *Monatshefte für Mathematik und Physik*, Vol. 36 (1929), p. 136.

The Real Unit Segment as a Number Field.

By E. T. Bell.

In constructing an arithmetic of symmetric functions, it became necessary to devise an algebra, analysis and arithmetic of the numbers in the closed interval [0, 1] isomorphic to the like, as commonly understood, for the closed interval $[-\infty, +\infty]$. As the solutions are of interest in themselves, one is given here independently of its applications.

1. Consider the real function $R(t)$,

$$R(t) = 1/2 + (1/\pi)\tan^{-1} t, \quad |\tan^{-1} t| \leqq \pi/2, \quad -\infty \leqq t \leqq +\infty.$$

Then the real function $R^{-1}(t)$ inverse to $R(t)$ is

$$R^{-1}(t) = -\cot \pi t, \qquad 0 \leqq t \leqq 1;$$

and

$$0 \leqq R(t) \leqq 1, \qquad -\infty \leqq R^{-1}(t) \leqq +\infty.$$

If a, b are any real numbers, $R(a+b)$, $R(a-b)$, $R(a\cdot b)$, $R(a \div b)$ are real numbers ($b \neq 0$ in the last).

We now define the operations $|+|$, $|\cdot|$ upon the numbers $R(t)$ by

$$R(a)\,|+|\,R(b) = R(a+b),$$
$$R(a)\,|\cdot|\,R(b) = R(a\cdot b).$$

Hence the inverse, $|-|$, of $|+|$, and the inverse, $|\div|$, of $|\cdot|$, are given by

$$R(a)\,|-|\,R(b) = R(a-b),$$
$$R(a)\,|\div|\,R(b) = R(a \div b),$$

in the last of which $b \neq 0$.

Write $R(a) = \alpha$, $R(b) = \beta$ for the moment. Then $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $a = R^{-1}(\alpha)$, $b = R^{-1}(\beta)$. Hence, from the definition of $|+|$, it follows that

$$\alpha\,|+|\,\beta = R[R^{-1}(\alpha) + R^{-1}(\beta)],$$
$$= 1/2 - (1/\pi)\tan^{-1}[\cot \pi\alpha + \cot \pi\beta],^*$$

and similarly for $|-|$, $|\cdot|$, $|\div|$.

The open interval derived from [0, 1] by omitting both end points 0, 1, will be denoted by {0, 1}, and likewise for $[-\infty, +\infty]$, $\{-\infty, +\infty\}$.

* The principal value of any inverse tangent is always to be understood; thus $-\pi/2 < \tan^{-1}\phi \leqq +\pi/2$.

THEOREM 1. *The set of all numbers in* $\{0, 1\}$ *is a field, say* $\{R\}$, *with respect to* $|+|$, $|-|$, $|\cdot|$, $|\div|$ (*addition, subtraction, multiplication, division in* $\{R\}$), *defined by*

$$a\,|+|\,b = 1/2 - (1/\pi)\tan^{-1}(\cot \pi a + \cot \pi b),$$
$$a\,|-|\,b = 1/2 - (1/\pi)\tan^{-1}(\cot \pi a - \cot \pi b),$$
$$a\,|\cdot|\,b = 1/2 + (1/\pi)\tan^{-1}(\cot \pi a \cot \pi b),$$
$$a\,|\div|\,b = 1/2 + (1/\pi)\tan^{-1}(\cot \pi a \tan \pi b),$$
$$0 < a < 1, \qquad 0 < b < 1.$$

The zero, $0'$, *and the unity,* $1'$, *of* $\{R\}$ *are*

$$0' = 1/2, \qquad 1' = 3/4.$$

The field $\{R\}$ *is isomorphic to the field of all numbers in* $\{-\infty, \infty\}$, *in which* $0, 1, +, -, \cdot, \div$ *have their usual meanings.*

THEOREM 2. *The ideal elements* $|+|\,\infty'$ *and* $|-|\,\infty'$ (*respectively the "positive infinity" and the "negative infinity" adjoined to* $\{R\}$), *having with respect to numbers and operations of* $\{R\}$ *the abstractly identical properties of* $+\infty$ *and* $-\infty$ *with respect to* $[-\infty, +\infty]$, *are*

$$|+|\,\infty' = 1, \qquad |-|\,\infty' = 0.$$

The result of adjoining $|+|\,\infty'$ and $|-|\,\infty'$ to $\{R\}$ will be denoted by $[R]$. The following makes Theorem 2 more explicit.

THEOREM 3. *The negative,* $|-|\,t$, *of any number* t *in* $[R]$ *is given by*

$$|-|\,t = 1 - t, \qquad 0 \leqq t \leqq 1;$$

the reciprocal, $1'\,|\div|\,t$, *is given by*

$$1'\,|\div|\,t = 1/2 - t, \qquad \epsilon \leqq t \leqq 1/2 - \epsilon,$$
$$1'\,|\div|\,t = 3/2 - t, \qquad 1/2 + \epsilon \leqq t \leqq 1 - \epsilon,$$

where ϵ *is a positive real variable* $< 1/2$ *which attains the limit* 0.

2. Serial order is introduced into $[R]$ by the definitions of the relations $|>|$, $|<|$. We shall say that the number t in $[R]$ is *positive* in $[R]$, written $t\,|>|\,0'$, if and only if $t > 1/2$; if $t < 1/2$, we shall say that t is *negative* in $[R]$, and write $t\,|<|\,0'$. If a, b are in $[R]$, and if $a \neq b$, we shall say that *a is greater than b* in $[R]$, and write $a\,|>|\,b$, if $(a\,|-|\,b)\,|>|\,0'$. Similarly, *a is less than b* in $[R]$, $a\,|<|\,b$, if $(a\,|-|\,b)\,|<|\,0'$. *Equality* in R is indicated by $=$, instead of $|=|$, as the meaning is that of $=$ for real numbers.

From the definition of $a \mid - \mid b$ and the inequalities

$$\begin{aligned} &\cot \pi a < \cot \pi b && \text{if } a > b, \\ &\cot \pi a > \cot \pi b && \text{if } a < b, \\ &0 \leqq a \leqq 1, && 0 \leqq b \leqq 1, \end{aligned}$$

we have the

THEOREM 4. *If a, b are unequal in $[R]$, $a \mid > \mid b$ or $a \mid < \mid b$ according as $a > b$ or $a < b$. Hence to analysis for $[-\infty, +\infty]$ there is an isomorph for $[0, 1]$ in $[R]$.*

3. The theory of numbers in $[R]$ will follow from the definitions of integer and arithmetical divisibility in $[R]$.

By definition, $0'$ $(=1/2)$ is a rational integer (*the zeroth rational integer*) in $[R]$. From $0'$ we define the nth *positive integer* n' in $[R]$ inductively by n successive additions in $[R]$ of the unity $1'$ $(=3/4)$ in $[R]$; the nth *negative,* or the $-n$th, *integer* in $[R]$ is then $\mid - \mid n'$. Hence the mth integer m' in $[R]$ is

$$m' = R(m) \qquad (m \text{ an integer } \gtreqless 0).$$

Let c', b' be integers in $[R]$. Then c' is said to be *divisible by b' in* $[R]$ if and only if $c' \mid \div \mid b'$ is an integer in $[R]$, and we write $b' \mid c'$ when c' is divisible by b' in $[R]$. Hence $b' \mid c'$ when and only when c is divisible by b. If c', $\mid \pm \mid 1'$ are the only values of b' for which $b' \mid c'$, c' is called a *prime in* $[R]$ if $c' \neq 1'$. The positive primes in $[R]$ are therefore the $R(p)$, where p runs through all positive rational primes. Unique decomposition of integers in $[R]$ into primes in $[R]$ follows, and it is obvious that the whole theory of the rational integers goes over in simple isomorphism into a theory of rational integers in $[R]$. The extension to algebraic numbers and ideals in $[R]$ is immediate.

Concerning Collections of Continua not All Bounded.*

By J. H. Roberts.

In his paper *Concerning upper semi-continuous collections of continua* † R. L. Moore proved that if G is any upper semi-continuous collection of mutually exclusive bounded continua which fills the plane and no one of which separates the plane then G is a plane with respect to its elements. Moore suggested to me the study of what kinds of spaces would be obtained by removing the condition of the *boundedness* of the continua of G. In particular he raised this question: *If G is an upper semi-continuous collection of mutually exclusive continua which fills a plane S and no continuum of G separates S, then is G topologically equivalent to a subset of the plane?*

In the present paper an example is given which shows that if some continua of such an upper semi-continuous collection G are unbounded then the space of elements of G is not necessarily even *metric*. An example is also given of an upper semi-continuous collection G of continua filling the plane, no one of which separates the plane, such that G is topologically equivalent to a sphere. In this example only one continuum of the collection G is unbounded. It is then shown that with the additional hypotheses that the space of elements of G is metric and that G contains more than one unbounded continuum the answer to Moore's question is in the affirmative. Indeed the particular subsets of a plane which are topologically equivalent to such spaces of elements are completely characterized.

Definition 1.‡ If G is a collection of mutually exclusive continua then G is said to be an *upper semi-continuous* collection if for every sequence of

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* Presented to the Society, Dec. 27, 1928.

† *Transactions of the American Mathematical Society*, Vol. 27 (1925), pp. 416-428.

‡ It is obvious that for the case where the continua of G are bounded, definitions 1 and 2 are equivalent to the following given by Moore (ibid.): "A collection G of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance from g less than d then the upper distance of x from g is less than e. The element p of such a collection G is said to be a *limit element* of the subcollection K of G if for every positive number e there exists some element of K which is distinct from p and whose upper distance from p is less than e."

continua $h_1, h_2, \cdots$ of G containing points $P_1, P_2, \cdots$ such that the sequence $P_1, P_2, \cdots$ has a sequential limit point P lying in a continuum g_P of G it is true that if $Q_1, Q_2, \cdots$ is a sequence of points such that for every i the point Q_i belongs to h_i then all the limit points of the point set $Q_1 + Q_2 + \cdots$ belong to the continuum g_P.

Definition 2. If G is an upper semi-continuous collection of continua then an element g of G is said to be a *limit element* of a set K of elements of G if and only if g contains a point P which is a limit point of the point set obtained by adding together all of the continua of K except g.

Example 1. Let r be a ray of an open curve. Let G be the collection of mutually exclusive continua filling the plane whose only non-degenerate * element is r. Clearly G is an upper semi-continuous collection. I will show that with respect to the above defined notion of limit element G is not a metric space. Let $C_1, C_2, C_3, \cdots$ be a set of concentric circles all containing points of the ray r, and such that for every n the diameter of C_n is n. For every n let L_n be an infinite sequence of points on C_n no one of which belongs to r, but such that some point of r is a sequential limit point of the sequence L_n.

Now suppose that G is metric. Then there exists a distance function $\delta(x, y)$ such that for every two distinct elements x and y of G we have $\delta(x, y) > 0$, and such that an element g of G is a limit element of a set K of elements if and only if for every positive number e the collection K contains an element k_e such that $\delta(k_e, g) < e$. Now for every n the element r is a limit element of the sum of the elements of the sequence L_n. Therefore for every n there exists an element p_n of L_n such that $\delta(r, p_n) < 1/n$. Then r is a limit element of the set of elements $p_1 + p_2 + p_3 + \cdots$. But no point of r is a limit point of the point set $p_1 + p_2 + p_3 + \cdots$. Hence the supposition that G is metric has led to a contradiction.

Example 2. (See fig. 1). Assume a rectangular coördinate system. If c is a real number ($c \geqq 2$) let g_c be the continuum consisting of the two intervals $x = \pm 1/c$, $0 \leqq y \leqq c$, together with the larger arc of the circle with the origin as center and radius equal to $[c^2 + 1/c^2]^{\frac{1}{2}}$ which has just its end points on these intervals. Let G be the collection of continua containing (a) the continuum g_c for every real number c ($c \geqq 2$), (b) the ray $x = 0$, $y \geqq 0$, and (c) every remaining *point* of the plane. It can be seen that G is an upper semi-continuous collection, and that it is metric with respect to

* An element is said to be *degenerate* if it contains but a single point. See R. L. Moore, "Concerning Upper Semi-Continuous Collections," *Monatshefte für Mathematik und Physik*, Vol. 36 (1929), pp. 81-88.

its elements. That G is not topologically equivalent to any subset of the plane follows from the fact that (by theorem III) G is topologically equivalent to a sphere.*

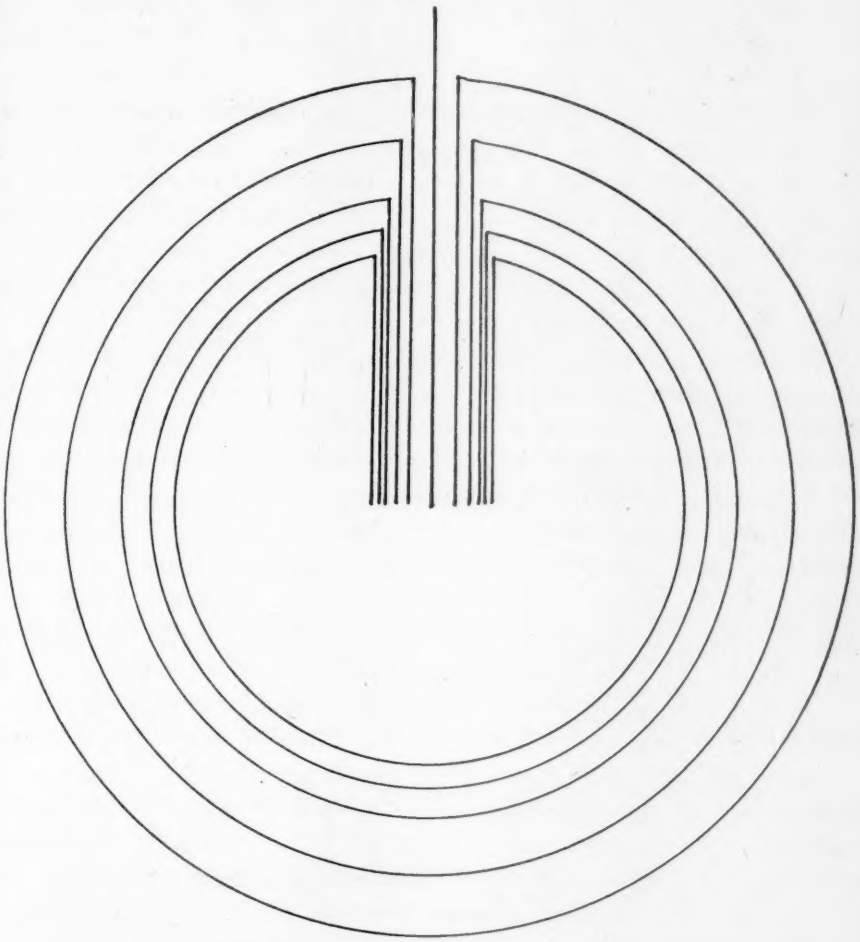

FIGURE 1.

THEOREM I.† *Let G be a collection of mutually exclusive unbounded*

* Another example is the following: Let G be a collection of mutually exclusive continua filling the plane of which the only non-degenerate element is a circle plus its exterior. Clearly G is topologically equivalent to a sphere. In the example given above no continuum of the collection G contains a domain.

† Dr. N. E. Rutt has obtained an independent proof of this theorem. In fact credit is due him for having been, as far as I know, the first to consider this proposition. Without knowing Dr. Rutt was working on this problem I obtained, as a proof of a somewhat different theorem, the argument herewith given.

continua whose sum is a closed point set M. If A and B are points and no continuum of G separates A from B then M does not separate A from B.

Proof. Suppose that the hypothesis of the theorem is satisfied but that M separates a point A from a point B. On the basis of this supposition I will show that M contains an *irreducible integral continuum* * which separates A from B.

In view of Mazurkiwicz's extension † of the Phragmen-Brouwer theorem to unbounded point sets it follows from Knaster and Kuratowski ‡ that every point set separating A from B contains a continuum separating A from B. In view of this and the additional fact that M is closed it is clear that some maximal connected subset of M is an *integral* continuum separating A from B. Let the interiors of the circles with rational radii and rational centers be ordered $R_1, R_2, \cdots$. For each positive integer n let H_n denote the sum of the continua of G which contain a point in R_n. Suppose that M does not contain an irreducible integral continuum separating A from B. Let n_1 be the smallest integer such that $M - H_{n_1}$ contains an integral continuum separating A from B, and let M_1 be such an integral continuum. Let n_2 be the smallest integer greater than n_1 such that $M_1 - M_1 \cdot H_{n_2}$ contains an integral subcontinuum separating A from B. It is clear that there exists a sequence of numbers $n_1, n_2, \cdots$ such that $n_1 < n_2 < n_3 < \cdots$ and a sequence of integral continua $M_1, M_2, M_3, \cdots$ such that for each k (a) the continuum M_k contains M_{k+1}, (b) M_k separates A from B and (c) M_k contains no point of $H_{n_1} + H_{n_2} + \cdots + H_{n_k}$ but if n is any integer less than n_k and is not any n_i $(i = 1, 2, \cdots, k-1)$ then every integral subcontinuum of M_{k-1} which separates A from B does contain a point of H_n. Now for every k every simple continuous arc from A to B contains a closed subset of M_k. Hence there exists at least one point common to all of the sets $M, M_1, M_2, \cdots$. Let Q denote the set of all points common to $M, M_1, M_2, \cdots$. Clearly Q is closed and is an integral subset of M. That Q separates A from B follows from the fact that Q contains a point on every arc from A to B. It follows from property (c) above that no integral continuum which is a proper subset of Q

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* By an *integral continuum* is meant a continuum N such that if g is a continuum of G and N contains a single point of g then N contains every point of g. An integral continuum N is said to be an *irreducible integral continuum* (with respect to the property of separating A from B) if N separates A from B but no integral continuum of M which is a proper subset of N separates A from B.

† "Extension du théorème du Phragmen-Brouwer aux ensembles non bornés," *Fundamenta Mathematicae*, Vol. 3 (1922), p. 20.

‡ "Sur les ensembles connexes," *Fundamenta Mathematicae*, Vol. 2 (1921), p. 233.

separates A from B. But by Knaster and Kuratowski the set Q contains a continuum separating A from B. Hence Q *is* an irreducible integral continuum separating A from B.

Now there exist three continua g_1, g_2, and g_3 of G which belong to Q and three arcs AP_1, AP_2, and AP_3 such that A is the common part of each two of these three arcs, P_i belongs to g_i $(i=1, 2, 3)$ and no other point of the arc AP_i belongs to Q. Let D_1 denote that complementary domain of $\Sigma_1^3 AP_i + g_i$ which contains B. One continuum of the set g_1, g_2, g_3 contains no point in $\bar{D}_1$. Hence the point set $Q \cdot \bar{D}_1$ is a *proper* integral subset of Q and therefore does not separate A from B. Let BA denote an arc from B to A which has no point in common with $Q \cdot \bar{D}_1$. Let C denote the first point of the boundary of D_1 on BA in the order from B to A. Since the arc BA has no point in common with $Q \cdot \bar{D}_1$ the point C must belong to one of the arcs AP_1, AP_2, or AP_3, and must be distinct from P_1, P_2, and P_3. Obviously then there exists an arc from B to A which contains no point whatsoever of Q. Hence the supposition that M separates A from B has led to a contradiction.

COROLLARY 1. *Suppose P is a point, G is a collection of continua each two having in common just the point P, and M, the sum of the continua of G, is closed and bounded. Then if A and B are points and no continuum of G separates A from B the continuum M does not separate A from B.*

COROLLARY 2. *Suppose G is an upper semi-continuous collection of mutually exclusive continua filling the plane. Let T be the sum of all the unbounded continua of G. Then if A and B are points which are not separated by any unbounded continuum of G they are not separated by T.*

THEOREM II. *Let G be an upper semi-continuous collection of unbounded continua lying in a plane S and let M be a continuum of elements of G. Let g_1, g_2, g_3 and g_4 be any elements of M no two of which are separated in S by an element of G. Then the sum of two of the elements g_2, g_3, g_4, disconnect M between the third one and g_1.*

Proof. There exists a point A, four arcs AP_1, AP_2, AP_3 and AP_4 such that A is the common part of each two, P_i belongs to g_i $(i=1, \cdots, 4)$ and no other point of AP_i belongs to any continuum g_j $(j=1, \cdots, 4)$. Two of the arcs AP_i $(i=2, 3, 4)$ plus the corresponding continua of the set g_2, g_3, g_4 separate the third one from g_1. Suppose that Z $(Z=g_2+g_3$

$+ AP_2 + AP_3$) separates g_1 from g_4. (See fig. 2). Let D_1 and D_4 denote the complementary domains of Z containing g_1 and g_4, respectively. Let g denote any continuum of G in M distinct from g_2 and g_3. Not both of the sets $D_1 \cdot g$ and $D_4 \cdot g$ are unbounded. For since g does not separate g_2 from g_3 there exists a simple continuous arc P_2CP_3 with no point in g_2 or g_3 except P_2 and P_3, and no point in g. Let J be the interior of a circle which encloses the arcs P_2CP_3 and P_2AP_3. It can be shown that the continuum $g_2 + g_3 + P_2CP_3$ separates all points of D_1 and D_4 not in J. But g is connected and has no point in common with this continuum. Hence g has points in only one of the sets $D_1 \cdot (S - \bar{J})$ and $D_4 \cdot (S - \bar{J})$.

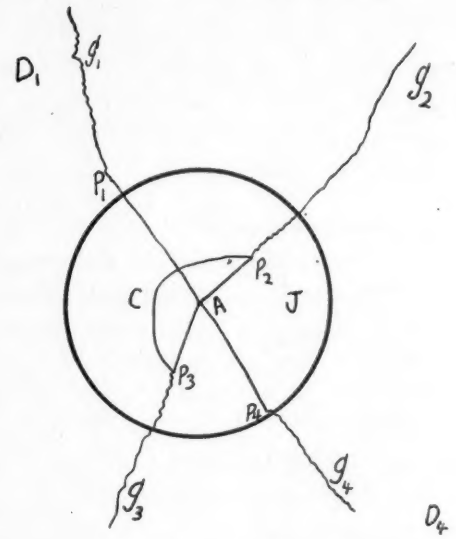

FIGURE 2.

Let S_1 denote the point set consisting of the sum of all continua of G in M which are unbounded in the domain D_1, and S_2 the sum of all those unbounded in D_4. The set S_1 contains g_1 and the set S_2 contains g_4. Suppose that a point E of S_1 is a limit point of S_2. Then there exists a sequence of continua $h_1, h_2, \cdots$ lying in S_2 containing a sequence of points $E_1, E_2, \cdots$ which has E as sequential limit point. If g_E denotes the continuum of G containing E, in view of the fact that G is upper semi-continuous, it is clear that $g_E \cdot D_4$ is unbounded. But g_E is in S_1. This contradiction shows that no point of S_1 is a limit point of S_2. Similarly no point of S_2 is a limit point of S_1. Hence g_1 and g_4 are separated in M by the sum of the continua g_2 and g_3.

LEMMA 1. *Suppose G is an upper semi-continuous collection of continua lying in a plane S and G is metric. Then if $\delta(x, y)$ denotes a distance function, for each element g of G there exists a circle C_g and a positive number e_g such that if x is an element of G for which $\delta(x, g) < e_g$ then x contains a point inside the circle C_g.*

Proof. The function $\delta(x, y)$ satisfies the following conditions: (1) $\delta(x, y) > 0$ if $x \neq y$, (2) $\delta(x, x) = 0$, (3) $\delta(x, y) + \delta(y, z) \geqq \delta(x, z)$, and (4) an element g of G is a limit element of a set K if and only if for every positive number e there exists an element g_e of K for which $\delta(g, g_e) < e$. Let P be a point of the plane, and for each positive integer n let C_n denote the circle with P as center and n as radius. Let g denote any element of G. Suppose that for every n there exists an element x_n of G such that $\delta(x_n, g) < 1/n$ and x_n has no point within the circle C_n. Then the limit of $\delta(x_n, g)$, as n increases indefinitely, is zero, whence g is a sequential limit element of the sequence $x_1, x_2, \cdots$. But this is impossible since no sequence of points $P_1, P_2, \cdots$ where for every n the point P_n belongs to x_n, has a sequential limit point. This contradiction proves the lemma.

THEOREM III. *If G is an upper semi-continuous collection of continua filling the plane S such that no continuum of G separates S and G is metric, then in order that G be topologically equivalent to a sphere it is necessary and sufficient that exactly one continuum of G be unbounded.*

Proof. The necessity of the condition follows from these two facts: (1) If *no* continuum of G is unbounded then G is * equivalent to the whole plane S. (2) If at least two continua of G are unbounded then G is † equivalent to a proper subset of the plane.

The condition is sufficient. Let g be the continuum of G which is unbounded. The point set $S - g$ is a simply connected domain and is therefore topologically equivalent to the plane S. Hence by Moore's theorem (ibid.) the collection $G - g$ ‡ is topologically equivalent to a plane and therefore to a sphere minus one point.

By lemma 1 there exists a circle C and a positive number e such that if x is an element of G and $\delta(x, g) < e$ then x has a point inside the circle C. Since G is upper semi-continuous and g is the only unbounded continuum of G it follows that if C_1 is any closed subset of C which does not contain

* R. L. Moore, *loc. cit.*

† This follows from theorem IV, the proof of which does not depend upon theorem III.

‡ That is, the set of all continua of G except g.

a point of g, and K is the sum of the elements of G which contain points of C_1, then K is a bounded point set. It therefore follows that every unbounded set of elements of $G - g$ has g as limit element. Hence G is topologically equivalent to a sphere.

LEMMA 2. *Let G be an upper semi-continuous collection of continua filling the plane, none separating the plane, such that G is metric. Then if T is the sum of all unbounded continua of G and M is a maximal connected subset of T it follows that no point of M is a limit point of $T - M$.*

Proof. Suppose that lemma 2 is not true. Then there exists an element g and a sequence of elements $g_1, g_2, \cdots$ with g as sequential limit element such that all of the continua $g, g_1, g_2, \cdots$ are unbounded and no two of them belong to the same component of T. By lemma 1 there exists a circle C and a positive number e such that if x is a continuum of G and $\delta(x, g) < e$ then x contains a point inside the circle C. With the help of the methods used in the proof of theorem II it can be shown that there exists a subsequence $g_{n_1}, g_{n_2}, \cdots$ of the sequence $g_1, g_2, \cdots$ and for each k an arc A_kB_k of the circle C having just A_k in g and B_k in g_{n_k}, and a domain D_k which is one of the complementary domains of $g + g_{n_k} + A_kB_k$, such that, T_k denoting the sum of all continua of G which are unbounded in D_k, the set T_k contains the set T_{k+1}. It can also be shown that if $\bar{e}$ is any positive number then there exists an integer m such that if x is an unbounded continuum of G which contains a point of C and $\delta(x, y) > \bar{e}$ then x does not belong to T_m.

Let m be an integer such that if x is a continuum of G which is a subset of T_m, and x contains a point of C, then $\delta(x, g) < e/2$. Since T_m contains continua of the sequence $g_1, g_2, \cdots$ and also contains g it follows that it is not connected. Therefore there exists an unbounded continuum r of G such that $T_m = H_g + H_r$, where H_g and H_r are mutually exclusive closed sets containing g and r, respectively. (See fig. 3). Since H_g and H_r are closed, and G is an upper semi-continuous collection, there exist two open sets R_g and R_r such that (1) $\bar{R}_g \cdot \bar{R}_r = 0$, (2) R_g contains $C \cdot H_g$ and R_r contains $C \cdot H_r$, and (3) if x is any continuum of G with a point in R_g $[R_r]$ then there exists a continuum k of G which belongs to H_g $[H_r]$ and has a point on C such that $\delta(x, k) < e/2$. Let P denote the center of the circle C and for each positive integer s let a_s be an arc lying in D_m except that its end points belong to g and r, respectively, and such that every point of a_s is at a distance greater than s from P. Since every continuum of the upper semi-continuous collection G which contains a point of C not in R_g or R_r is bounded in the domain D_m and since moreover the sum of all such continua is closed

(R_g and R_r being open) it follows that there exists an integer t such that if $s > t$ then no continuum of G which contains a point of a_s can contain a point of C not belonging to $R_g + R_r$.

For every integer s greater than t let Q_s denote the first point of a_s in the order from r to g which belongs to a continuum of G which contains a point of $\bar{R}_g$. Let x_s denote that continuum of G which contains Q_s. Since $\delta(x_s, g) < e$, there exists a point P_s of a_s which preceeds Q_s in the order from r to g, such that if y_s is any continuum of G which contains a point of the

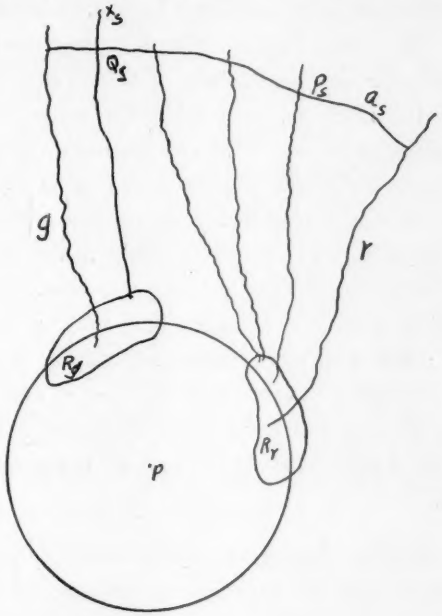

FIGURE 3.

arc P_sQ_s, then $\delta(x_s, g) < e$, whence y_s contains a point of $\bar{R}_g + \bar{R}_r$. But if y_s is different from x_s it does not contain a point in $\bar{R}_g$. Therefore there exists a sequence of continua of G, $h_1, h_2, \cdots$ with x_s as sequential limit element and such that for every integer k the continuum h_k contains a point in $\bar{R}_r$. Therefore, since G is upper semi-continuous, x_s contains a point of both $\bar{R}_g$ and $\bar{R}_r$. Since for every s $(s > t)$ the continuum x_s has a point in the closed and bounded set $\bar{R}_g$ it follows that there exists a continuum z of G which is a limit element of the sum of the elements of the sequence x_{t+1}, $x_{t+2}, \cdots$. Obviously z is unbounded in $\bar{D}_m$. Moreover it contains a point both of $\bar{R}_g$ and of $\bar{R}_r$. But this is contrary to the fact that T_m, the sum of all continua of G which are unbounded in $\bar{D}_m$, is the sum of the mutually separated sets H_g and H_r, where H_g contains no point of $\bar{R}_r$ and H_r contains

no point of $\bar{R}_g$. Hence our supposition that lemma 2 is false has led to a contradiction.

LEMMA 3. *If G is an upper semi-continuous collection of continua which fills the plane, no continuum of G separates the plane but at least two continua of G are unbounded, and G is metric, then no element of G is a component of the sum of all the unbounded continua of G.*

Proof. Suppose that g is an unbounded continuum of G. Let k be any other unbounded continuum of G and let $\bar{e}$ be $\delta(g, k)$. Let e be any positive number less than $\bar{e}$. By lemma 1 there exists a circle C and a positive number e_1 $(e_1 < e)$ such that if x is a continuum of G and $\delta(g, x) < e_1$ then x contains a point inside the circle C. Let P be the center of C and for every positive integer n let a_n be an arc with end points on g and k, respectively, every point of which is at a distance greater than n from P. Since $\delta(g, k) > e_1$ it follows that there exists a continuum x_n of G containing a point of a_n and such that $e_1/2 < \delta(x_n, g) < e_1$. For every n the continuum x_n has a point within C and a point at a distance greater than n from P. Hence if for each n the continuum x_n is bounded the sum of the elements of the sequence $x_1, x_2, \cdots$ has a limit element x_e, which is unbounded. Since $e_1/2 < \delta(x_n, g) < e_1$ it follows that $\delta(x_e, g) \leqq e_1 < e$, and x_e is distinct from g. Hence for every positive number e there exists an unbounded continuum x_e of G distinct from g such that $\delta(x_e, g) < e$. Lemma 3 therefore follows in view of lemma 2.

LEMMA 4. *Under the hypotheses of lemma 3 if M is a collection of continua of G whose sum is a component of the sum of all unbounded continua of G then M is a simple closed curve or an open curve.*

Proof. It readily follows from theorem II that M is an atriodic * continuous curve. Hence M is an open curve, a simple closed curve, an arc, or a ray of an open curve. Let g denote any element of M. Let g_1, g_2 and g_3 denote three other elements of M such that the sum of g and g_2 separates g_1 from g_3 in M. Let AB denote an arc having only A in common with g and only B in common with g_2. Let D be either of the two complementary domains of $g + g_2 + AB$. By a slight modification of the proof of lemma 3

* If O, A_1, A_2 and A_3 are distinct points and OA_1, OA_2 and OA_3 are arcs each two having only O in common, then the continuum $OA_1 + OA_2 + OA_3$ is called a *triod*. A continuous curve which does not contain a triod is said to be *atriodic*. See R. L. Moore, "Concerning Triods in the Plane and the Junction Points of Plane Continua," *Proceedings of the National Academy of Sciences*, Vol. 14 (1928), pp. 85-88.

it can be seen that g is a limit element of the elements of M which are unbounded in D. Therefore g is not an end element of M. Hence M contains no end elements and is therefore a simple closed curve or an open curve.

THEOREM IV. *If G is an upper semi-continuous collection of continua which fills a plane S and is metric, at least two continua of G are unbounded and no one of them separates S, then G is topologically equivalent to a domain D plus its boundary where* (a) *D is the interior of a circle, or* (b) *D is an unbounded domain whose boundary consists of one or more open curves no one containing a limit point of the sum of the others.*

Proof. Case 1. Suppose first that some collection M of unbounded continua of G is a simple closed curve. Suppose g is any element of M and h is any other unbounded continuum of G. Since g is not an end element of M it follows with the help of the methods used to prove theorem II that $g+h$ separates $M+h$. Since no element of M separates M it follows that h belongs to M; that is, every unbounded continuum of G is an element of M. Let P be any point and for every positive integer n let C_n be the circle with P as center and radius n. From lemma 1 it follows that for any element g of M there exists a positive integer k_g such that if x is a continuum of G and $\delta(x, g) < 1/k_g$, then x contains a point inside the circle C_{k_g}. Since moreover every infinite set of elements of M has a limit element (M being a simple closed curve) it follows that there exists a single positive integer k such that if g is any element of M and x is a continuum of G such that $\delta(x, g) < 1/k$, then x contains a point inside the circle C_k.

Let $g_1, g_2, \cdots$ denote any infinite sequence of distinct elements of G. If for any n all but a finite number of these continua contain a point inside C_n then the set of elements $g_1 + g_2 + \cdots$ has a limit element. Otherwise for every n $(n > k)$ let r_n be an integer such that g_{r_n} has no point inside C_n. Obviously $\delta(g_{r_n}, g) \geqq 1/k$ for every element g of M. It follows that there exists a continuum h_n of G such that $1/2k < \delta(h_n, g)$ for every element g of M, but $\delta(g, h_n) < 1/k$ for some element g of M, and h_n contains a point outside C_n. The sequence $h_{k+1}, h_{k+2}, \cdots$ obviously contains a subsequence with a sequential limit element h. Clearly h is unbounded. But since $1/2k < \delta(h_n, g)$ for every unbounded continuum g of G we have $1/2k \leqq \delta(h, h)$. This contradiction shows that for some integer n every element of G has a point inside the circle C_n so that G is compact.

Now every element of M is a limit element of the set of elements $G - M$. The sum of all the continua of $G - M$ is a connected domain D with a connected boundary. Hence D is topologically equivalent to the whole plane

S, and by Moore's theorem referred to above the collection $G - M$ is topologically equivalent to the plane S, or to the interior of a circle. Its boundary is equivalent to a circle. This completes the proof for case 1.

Case 2. Suppose that no collection of unbounded continua of G is a simple closed curve. Let T be the sum of all unbounded continua of G and let $M_1, M_2, \cdots$ be the subcollections of continua of G, the sums of whose continua form the components of T. In view of theorem I it can be seen that $S - T$ is a simply connected domain D. Hence in view of Moore's theorem (ibid.) the collection $G - (M_1 + M_2 + \cdots)$ is a simply connected domain. Since in addition for every i every element of M_i is a limit element of $G - M_i$ theorem IV is established.

THEOREM V. *If K is a point set described in the conclusion of theorem IV then there exists an upper semi-continuous collection G of continua which fills the plane such that* (1) *no continuum of G separates the plane and* (2) *with respect to its elements G is topologically equivalent to the point set K.*

Proof. For the case where K is a circle C plus its interior let the elements of G be the *points* interior to C together with the subsets of all rays starting at the center of C which lie on and outside of C. Obviously G is metric, and is topologically equivalent to C plus its interior. Suppose K is a domain D plus its boundary, where D is a connected domain bounded by open curves $N_1, N_2, \cdots$, such that for each i no point of N_i is a limit point of the sum of the other open curves of the sequence $N_1, N_2, \cdots$. For each i let D_i be the complementary domain of N_i which does not contain D. It can be shown that there exists an upper semi-continuous set G_i of rays of open curves lying in $\bar{D}_i$, each starting on N_i, and such that every point of $\bar{D}_i$ belongs to some ray of the set G_i. Let G be the set of all points of D together with all continua of the collection G_i for every i. The set G_i is topologically equivalent to the open curve N_i. It therefore follows that the collection G satisfies the conclusion of the theorem.

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Limited Tri-Linear Forms in Hilbert Space.*

By William L. Hart.

1. *Introduction.* At the foundation of Hilbert's theory of functions of infinitely many variables we find the notion of a limited † function, and, in particular, the idea of a limited bi-linear form in infinitely many variables. Similarly, one may consider limited tri-linear forms, or, in general, limited m-linear forms where m is a positive integer. The most fundamental properties of limited m-linear forms are easily anticipated by analogy with bi-linear forms. For a given $m > 2$, the proofs of these properties can in general be developed along the same lines as the existing demonstrations of results for bi-linear forms.‡ In this connection, however, one important exception is to be noted; the vital theorem on the uniform finiteness of a bi-linear form, which converges over the unit sphere in Hilbert space, has been established § by a method which demands essential modification in order to apply to the case of an m-linear form with $m > 2$.

The main purpose of this paper is to demonstrate the theorem on uniform finiteness for tri-linear forms by use of a method which applies as well in the consideration of m-linear forms, for any $m \geqq 2$. It is emphasized that this method, which is presented in Section 3, has merit solely because of its applicability when $m > 2$; its use when $m = 2$ would lead to much more complication than is found in the classical proof for this case. As a minor part of the paper, in Section 2 there is given a summary of other properties of tri-linear forms with details of the proofs only in a few instances where there is novelty as compared with the bi-linear case. In the entire discussion, all number symbols are supposed to be real valued.

2. *Limited Tri-linear Forms.* In the future, any small greek letter will denote a vector with infinitely many coördinates. Thus $\xi = (x_1, x_2, \cdots)$. By the n-th *section* of a vector ξ we shall mean the vector $\xi^{(n)} = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$, where all coördinates beyond the n-th one are zeros. If ξ is in

* Presented to the American Mathematical Society, June 20, 1929.

† David Hilbert, *Grundzüge Einer Allgemeinen Theorie der Linearen Integralgleichungen* (B. G. Teubner), p. IV.

‡ For definitions and theorems relating to limited linear and bi-linear forms, see Hellinger-Toeplitz, *Mathematische Annalen*, Vol. 69 (1910), p. 289.

§ Hellinger-Toeplitz, *loc. cit.*, p. 321; the details at the top of p. 324 do not generalize to the case of m-linear forms, with $m > 2$.

Hilbert space, the positive square root of the convergent series $\sum_{i=1}^{\infty} x_i^2$ will be denoted by $M(\xi)$ and will be called the modulus of ξ. We shall let H represent Hilbert space, and H_1 the set of vectors ξ where $M(\xi) \leqq 1$.

In accordance with Hilbert's definition of a limited function, we shall say that a three dimensional array $A \equiv (a_{ijk})$, and also the corresponding purely formal tri-linear expression

$$A(\xi, \rho, \sigma) = \sum_{i,j,k=1}^{\infty} a_{ijk} x_i r_j s_k, \tag{1}$$

is *limited,* in case there exists a bound $E > 0$ such that for all ξ, ρ, and σ in H, and for all integers h, m, and n,

$$| A(\xi^{(h)}, \rho^{(m)}, \sigma^{(n)}) | \leqq EM(\xi)M(\rho)M(\sigma). \tag{2}$$

An equivalent definition would result if (2) were altered by letting $h = m = n$. For abbreviation, we shall say that A is E-limited if (2) holds.

If any subscript of a_{ijk} is held fast, while the other two vary independently from 1 to ∞, we obtain a component matrix of A. If two subscripts are held fast, while the third varies from 1 to ∞, we obtain a component vector. The following Property I is easily proved by use of (2) and the corresponding definitions of limited linear and bi-linear forms; Properties II to IV can be demonstrated by direct methods like those used in similar connections in the consideration of bi-linear forms.

Property I. If A is E-limited, then each component vector, and each component matrix of A is E-limited.

Property II. If A is E-limited, then, if ρ and σ are in H, $\sum_{j,k=1}^{\infty} a_{ijk} r_j s_k$ converges uniformly with respect to the index i.

Property III. If A is E-limited, then, for all ρ and σ in H, the vector $(\sum_{j,k=1}^{\infty} a_{ijk} r_j s_k)$ lies in H and its modulus is at most $EM(\rho)M(\sigma)$.

Property IV. If A is E-limited, then, for every σ in H, the matrix $(\sum_{k=1}^{\infty} a_{ijk} s_k)$ is limited, with $EM(\sigma)$ as a bound.

We can consider evaluating a tri-linear form $A(\xi, \rho, \sigma)$ in 13 different ways, either as a triple series, or as an iterated series in twelve different ways.

Property V. If any one of the twelve iterated series for $A(\xi, \rho, \sigma)$ converges for all (ξ, ρ, σ) in H, then all of these twelve series converge and have the same sum for all (ξ, ρ, σ) in H.

Recognize that in the statement of Property V it is not assumed that A is limited. Let us discuss only the special case where we suppose that

$$W(\xi, \rho, \sigma) \equiv \sum_{i=1}^{\infty} x_i \sum_{j,k=1}^{\infty} a_{ijk} r_j s_k \tag{3}$$

converges for all (ξ, ρ, σ) in H. This implies that $\sum_{j,k}$, on the right in (3), converges for all (ρ, σ) in H. Therefore, by virtue of the property of uniform finiteness for a bi-linear form, it follows that, supposing i to be fixed, the matrix (a_{ijk}) is limited. Hence, the double sum $\sum_{j,k}$ can be changed to either possible iterated form. Thus, $A(\xi, \rho, \sigma)$ converges to $W(\xi, \rho, \sigma)$ under each of the following methods of iterated summation: $\sum_i[\sum_j(\sum_k)]$; $\sum_i[\sum_k(\sum_j)]$. Since the last method converges for all (ξ, σ) in H when ρ is held fast, it follows from the uniform finiteness of a bi-linear form that the matrix $(\sum_{j=1}^{\infty} a_{ijk} r_j)$ is limited. Then, starting from $\sum_i[\sum_k(\sum_j)]$, the fundamental convergence properties of a limited bi-linear form show that $A(\xi, \rho, \sigma)$ converges to $W(\xi, \rho, \sigma)$ under each of the following methods of iteration: $\sum_{i,k}(\sum_j)$; $\sum_k[\sum_i(\sum_j)]$. In this fashion, we show that all of the 12 iterated forms for $A(\xi, \rho, \sigma)$ converge to $W(\xi, \rho, \sigma)$.

Property VI. If A is E-limited, then, for all (ξ, ρ, σ) in H, the formal series for $A(\xi, \rho, \sigma)$ converges to an unique value when evaluated in any of the thirteen possible ways, and

$$|A(\xi, \rho, \sigma)| \leqq EM(\xi)M(\rho)M(\sigma). \tag{4}$$

The methods used in establishing the convergence properties of a limited bi-linear form would easily demonstrate that, if A is E-limited, then the particular iterated series $W(\xi, \rho, \sigma)$ of (3) converges and

$$|W(\xi, \rho, \sigma)| \leqq EM(\xi)M(\rho)M(\sigma). \tag{5}$$

Hence, it follows from Property V that (4) is true if $A(\xi, \rho, \sigma)$ is evaluated as an iterated series in any of the 12 possible ways. To complete the proof of Property VI, it would be necessary to show that $A(\xi, \rho, \sigma)$ converges to $W(\xi, \rho, \sigma)$ when evaluated as a triple series; that is, it should be proved that

$$\lim_{n\to\infty} \sum_{i,j,k=1}^{n} a_{ijk} x_i r_j s_k = \sum_{i=1}^{\infty} x_i \sum_{j,k=1}^{\infty} a_{ijk} r_j s_k, \tag{6}$$

where the right member, $W(\xi, \rho, \sigma)$, is known to converge. We omit the proof of (6), which is easily accomplished by the aid of (5).

Three successive applications of the Schwarz inequality * lead to the

* Hellinger-Toeplitz, *loc. cit.*, p. 293.

following result, which corresponds to a well known property of bi-linear forms.

Property VII. If the expression $V \equiv \sum_{i,j,k=1}^{\infty} a^2_{ijk}$ *converges, then* A *is limited, with* $V^{\frac{1}{2}}$ *as a bound.*

By reference to the definitions of continuity † which apply to functions defined in Hilbert space, and by use of the preceding properties of tri-linear forms, standard methods suffice to establish the following facts:

Property VIII. If $A(\xi, \rho, \sigma)$ *is limited, then it is continuous simultaneously in its arguments for all points in the region* S *defined by the following inequalities, where* g_1, g_2, *and* g_3 *are any assigned positive numbers:*

$$M(\xi) \leqq g_1; \qquad M(\rho) \leqq g_2; \qquad M(\sigma) \leqq g_3. \tag{7}$$

If $\sum_{i,j,k=1}^{\infty} a^2_{ijk}$ *converges, then* $A(\xi, \rho, \sigma)$ *is completely continuous in its arguments for all points* (ξ, ρ, σ) *satisfying* (7).

We shall call $C(\xi) \equiv \sum_{i,j,k=1}^{\infty} c_{ijk} x_i x_j x_k$ a cubic form when c_{ijk} is a symmetric function of (i, j, k). A cubic form is said to be limited, with E as a bound, in case

$$| C(\xi^{(n)}) | \leqq E M^3(\xi), \tag{8}$$

for every ξ in H and for every n.

Property IX. A necessary and sufficient condition that $C(\xi)$ *be limited is that the corresponding tri-linear form* $A(\xi, \rho, \sigma) \equiv \sum_{i,j,k=1}^{\infty} c_{ijk} x_i r_i s_k$ *be limited.*

It is obvious that, if $A(\xi, \rho, \sigma)$ is limited, then $C(\xi)$ is limited. To prove the converse, we note that the following identity is true:

$$\begin{aligned} 24A(\xi^{(n)}, \rho^{(n)}, \sigma^{(n)}) \equiv\ & [C(\xi^{(n)} + \rho^{(n)} + \sigma^{(n)}) - C(\xi^{(n)} - \rho^{(n)} + \sigma^{(n)}) \\ & - C(\xi^{(n)} + \rho^{(n)} - \sigma^{(n)}) + C(\xi^{(n)} - \rho^{(n)} - \sigma^{(n)})]. \end{aligned} \tag{9}$$

This identity can be verified by expanding the finite triple sums occurring, combining terms, and recalling that c_{ijk} is a symmetric function of (i, j, k). From (9) we see that, for all (ξ, ρ, σ) whose moduli are at most 1,

$$| A(\xi^{(n)}, \rho^{(n)}, \sigma^{(n)}) | \leqq 108E/24, \tag{10}$$

† Cf. Hellinger-Toeplitz, *loc. cit.*, p. 307-308.

if C is E-limited. Consequently, A is limited, with $36E/8$ as a bound.

3. *Uniform Finiteness of a Tri-Linear Form.*

Property X. If any one of the thirteen methods for evaluating a tri-linear form $A(\xi, \rho, \sigma)$ converges for all (ξ, ρ, σ) in H, then A is limited.

The case of Property X where we assume that $A(\xi, \rho, \sigma)$ converges as a triple series, for all (ξ, ρ, σ) in H, can be treated by essentially the same simple process which serves to prove the corresponding result for a bi-linear form, which is presumed to converge as a *double* series. Presuming that the instance of the triple series has been treated, it follows from Property V that we can complete the proof of Property X by discussing merely the case where the special iterated series $W(\xi, \rho, \sigma)$ of (3) is assumed to converge for all (ξ, ρ, σ) in H. Under this hypothesis, to show that A is limited, it will be sufficient to prove that $| W(\xi, \rho, \sigma) |$ is bounded in H_1, because

$$| \sum_{i,j,k=1}^{n} a_{ijk} x_i r_j s_k | = | W(\xi^{(n)}, \rho^{(n)}, \sigma^{(n)}) | . \tag{11}$$

To establish the desired result by an indirect process, let us suppose that $W(\xi, \rho, \sigma)$ is *not* bounded in H_1; it follows that there exists an infinite sequence of points $(\theta_p, \phi_p, \psi_p)$ in H_1 such that $\lim_{p\to\infty} | W(\theta_p, \phi_p, \psi_p) | = +\infty$. Let us agree to change the signs of all coördinates of those vectors θ_p for which $W(\theta_p, \phi_p, \psi_p) < 0$; then, $W(\theta_p, \phi_p, \psi_p) \geqq 0$ for all p and

$$\lim_{p\to\infty} W(\theta_p, \phi_p, \psi_p) = +\infty . \tag{12}$$

We shall proceed to a contradiction by defining vectors (η, ζ, τ) in H_1 for which $W(\eta, \zeta, \tau)$ does not converge.*

For future use, recall that, in the discussion of Property V, it was seen that, if $W(\xi, \rho, \sigma)$ converges for all (ξ, ρ, σ) in H, then each component matrix of A is limited.

LEMMA 1. *If m and n are assigned integers > 0, there exists a $d > 0$, where $d \leqq 1$, such that, if (ξ, ρ, σ) are points in H_1 for which $M[\rho - \rho^{(m)}] \leqq d$ and $M[\sigma - \sigma^{(m)}] \leqq d$, then*

$$| W(\xi^{(n)}, \rho, \sigma) - W(\xi^{(n)}, \rho^{(m)}, \sigma^{(m)}) | \leqq 1. \tag{13}$$

On applying the Schwarz inequality with respect to i in $W(\xi^{(n)}, \rho, \sigma)$ as given by (3), we obtain

* The same notion is the basis of the proof given by Hellinger-Toeplitz, *loc. cit.*, p. 321, in establishing the uniform finiteness of a bi-linear form.

$$(14) \qquad |W(\xi^{(n)}, \rho, \sigma)|^2 \leqq \left(\sum_{i=1}^{n} x_i^2\right) \sum_{i=1}^{n} \left(\sum_{j,k=1}^{\infty} a_{ijk} r_j s_k\right)^2.$$

Let K be a common bound for the n limited matrices $A_1, \cdots, A_n$ where $A_i \equiv (a_{ijk})_{j,k=1,2,\ldots}$. Then, if ξ is in H_1, from (14) we obtain

$$(15) \qquad |W(\xi^{(n)}, \rho, \sigma)| \leqq Kn^{\frac{1}{2}} M(\rho) M(\sigma).$$

We employ (15) twice in the following reduction:

$$\begin{aligned} &|W(\xi^{(n)}, \rho, \sigma) - W(\xi^{(n)}, \rho^{(m)}, \sigma^{(m)})| \\ &\quad \leqq |W(\xi^{(n)}, \rho - \rho^{(m)}, \sigma)| + |W(\xi^{(n)}, \rho^{(m)}, \sigma - \sigma^{(m)})| \\ &\quad \leqq Kn^{\frac{1}{2}} [M(\sigma) M(\rho - \rho^{(m)}) + M(\rho^{(m)}) M(\sigma - \sigma^{(m)})]. \end{aligned}$$

Hence, the statement of Lemma 1 is satisfied if d is chosen as the minimum of 1 and $1/2Kn^{\frac{1}{2}}$.

To proceed with the construction of (η, ζ, τ), select a sequence (c_{1j}) where $c_{1j} > 0$ and $\sum_{j=1}^{\infty} c^2{}_{1j} \leqq 1$. On account of (12), there exist vectors $(\alpha_1, \beta_1, \gamma_1)$ in H_1 such that $W(\alpha_1, \beta_1, \gamma_1) > 2/c^3{}_{11}$. Let $\xi_1 = c_{11}\alpha_1$, $\rho_1 = c_{11}\beta_1$, and $\sigma_1 = c_{11}\gamma_1$: then, $(\xi_1, \rho_1, \sigma_1)$ are in H_1 and $W(\xi_1, \rho_1, \sigma_1) > 2$. Therefore, there exist integers $m_1 > 0$ and $n_1 > 0$ such that $W(\xi_1{}^{(n_1)}, \rho_1{}^{(m_1)}, \sigma_1{}^{(m_1)}) > 2$. Now, let us define a block of the coördinates (y_i, z_j, t_j) of (η, ζ, τ) as follows, where (x_{i1}, r_{j1}, s_{j1}) are, respectively, coördinates of $(\xi_1, \rho_1, \sigma_1)$:

$$(16) \quad y_i = x_{i1}, \ (i = 1, 2, \cdots, n_1); \ z_j = r_{j1} \text{ and } t_j = s_{j1}, \ (j = 1, 2, \cdots, m_1).$$

We note that $M(\eta^{(n_1)}) \leqq c_{11} \leqq 1$; $M(\zeta^{(m_1)}) \leqq 1$; $M(\tau^{(m_1)}) \leqq 1$;

$$(17) \qquad W(\eta^{(n_1)}, \zeta^{(m_1)}, \tau^{(m_1)}) > 2.$$

Let $d = d_1$ correspond to $m = m_1$ and $n = n_1$ as stated in Lemma 1. It follows that, if (η, ζ, τ) are eventually defined so that, in addition to satisfying (16), $M(\zeta - \zeta^{(m_1)}) \leqq d_1$ and $M(\tau - \tau^{(m_1)}) \leqq d_1$, then

$$(18) \qquad W(\eta^{(n_1)}, \zeta, \tau) > 1.$$

At this point we have completed the first stage of the definition of (η, ζ, τ). To proceed with the second stage, we define $c_{2j} = d_1 c_{1j}$. Since each component matrix of A is limited, there exist bounds $R_1 > 0$, $F_1 > 0$, and $G_1 > 0$ such that, if (ξ, ρ, σ) are in H_1, then

$$(19) \quad |W(\xi^{(n_1)}, \rho, \sigma)| \leqq R_1; \ |W(\xi, \rho^{(m_1)}, \sigma)| \leqq F_1; \ |W(\xi, \rho, \sigma^{(m_1)})| \leqq G_1.$$

In view of (12), there exist vectors $(\alpha_2, \beta_2, \gamma_2)$ in H_1 such that

$$W(\alpha_2, \beta_2, \gamma_2) > (3 + 2R_1 + 2F_1 + 2G_1)/c^3{}_{22}.$$

Let $\xi_2 = c_{22}\alpha_2$; $\rho_2 = c_{22}\beta_2$; $\sigma_2 = c_{22}\gamma_2$; then, there exist integers $m_2 > m_1$ and $n_2 > n_1$ such that

$$W(\xi_2^{(n_2)}, \rho_2^{(m_2)}, \sigma_2^{(m_2)}) > 3 + 2R_1 + 2F_1 + 2G_1. \tag{20}$$

Let us extend the definition of (η, ζ, τ) as follows, where (x_{i2}, r_{j2}, s_{j2}) are, respectively, coördinates of $(\xi_2, \rho_2, \sigma_2)$:

$$y_i = x_{i2},\ (n_1 < i \leqq n_2); \quad z_j = r_{j2},\ t_j = s_{j2},\ (m_1 < j \leqq m_2). \tag{21}$$

We note that $M(\eta^{(n_2)}) \leqq (c^2_{11} + c^2_{22})^{1/2} \leqq (c^2_{11} + c^2_{12})^{1/2} \leqq 1$. Similarly, $M(\zeta^{(m_2)}) \leqq 1$ and $M(\tau^{(m_2)}) \leqq 1$. Let $P_{ijk} = a_{ijk}(y_i z_j t_k - x_{i2} r_{j2} s_{k2})$. Then, we verify that

$$W(\eta^{(n_2)}, \zeta^{(m_2)}, \tau^{(m_2)}) - W(\xi_2^{(n_2)}, \rho_2^{(m_2)}, \sigma_2^{(m_2)}) \tag{22}$$

$$= \sum_{i=1}^{n_1} \sum_{j,k=1}^{m_2} P_{ijk} + \sum_{i=n_1+1}^{n_2} \sum_{j=1}^{m_1} \sum_{k=1}^{m_2} P_{ijk} + \sum_{i=n_1+1}^{n_2} \sum_{j=m_1+1}^{m_2} \sum_{k=1}^{m_1} P_{ijk}.$$

From the definitions of R_1, F_1, and G_1, it is seen that the absolute value of the right member of (22) is at most $(2R_1 + 2F_1 + 2G_1)$. Hence, by use of (20) it follows from (22) that

$$W(\eta^{(n_2)}, \zeta^{(m_2)}, \tau^{(m_2)}) > 3. \tag{23}$$

Let $d = d_2$ correspond to $m = m_2$ and $n = n_2$ thru Lemma 1. It follows that, if (η, ζ, τ) are defined as points in H_1 subject to (16), (21), and to the conditions $M(\zeta - \zeta^{(m_2)}) \leqq d_2$ and $M(\tau - \tau^{(m_2)}) \leqq d_2$, then

$$W(\eta^{(n_2)}, \zeta, \tau) > 2. \tag{24}$$

We have now completed the second stage of the definition of (η, ζ, τ). Each succeeding stage depends on the preceding one as the second depended on the first. For instance, to commence the third stage, we define $c_{3j} = d_2 c_{2j}$; R_2, F_2, and G_2, etc. Then, there exist vectors $(\alpha_3, \beta_3, \gamma_3)$ in H_1 such that

$$W(\alpha_3, \beta_3, \gamma_3) > (4 + 2R_2 + 2F_2 + 2G_2)/c^3_{33};$$

we define $(\xi_3, \rho_3, \sigma_3)$, and integers $m_3 > m_2$ and $n_3 > n_2$. We extend the definition of (η, ζ, τ), and obtain a $d_3 \leqq 1$ from Lemma 1 so that, finally

$$W(\eta^{(n_3)}, \zeta, \tau) > 3, \tag{25}$$

provided that ζ and τ are eventually defined so that $M(\zeta - \zeta^{(m_3)}) \leqq d_3$ and $M(\tau - \tau^{(m_3)}) \leqq d_3$. The continuation of this process thru its successive stages yields a sequence of values (n_i, m_i, d_i) and a well determined set of coördi-

nates for (η, ζ, τ). We remark that these vectors are in H_1 because, at the i-th stage of their definition, we have

$$M(\eta^{(n_i)}) \leqq (c^2_{11} + c^2_{22} + \cdots + c^2_{ii})^{\frac{1}{2}} \leqq (c^2_{11} + c^2_{12} + \cdots + c^2_{1i})^{\frac{1}{2}} \leqq 1,$$

and, similarly, $M(\zeta^{(m_i)}) \leqq 1$ and $M(\tau^{(m_i)}) \leqq 1$. Moreover, for every i,

$$M^2(\zeta - \zeta^{(m_i)}) \leqq \sum_{j=i+1}^{\infty} M^2(\xi_j) \leqq \sum_{j=i+1}^{\infty} c^2_{jj} \leqq d_i^2 \sum_{j=i+1}^{\infty} c^2_{ij} \leqq d_i^2,$$

and, similarly, $M(\tau - \tau^{(m_i)}) \leqq d_i$. Consequently, the successive inequalities (18), (24), (25), etc., are valid:

$$W(\eta^{(n_i)}, \zeta, \tau) > i \qquad (i = 1, 2, \cdots,).$$

Hence, $W(\eta, \zeta, \tau)$ diverges. This contradicts our original hypothesis concerning $W(\xi, \rho, \sigma)$ and completes the proof of Property X.

If we assume that a complete theory for $(m-1)$-linear forms has been developed, we could establish the analogue of Property X for m-linear forms by use of the method just employed for tri-linear forms. In the discussion of the m-linear case we would refer to $(m-1)$-linear forms in the places occupied by bi-linear forms in the proof of Property X in this paper.

UNIVERSITY OF MINNESOTA,
FEBRUARY 15, 1930.

The Constants of the Disturbing Function.

By K. P. Williams.

1. Newcomb's development of the disturbing function requires the calculation of quantities $D^m c_n^{(i)}$, where

$$D = s(d/ds), \qquad c_n^{(i)} = s^{(n-1)/2} b_n^{(i)},$$

and the $b_n^{(i)}$ are the coefficients in the expansion

$$(1 + s^2 - 2s\cos\theta)^{-n/2} = (1/2)b_n^{(0)} + b_n^{(1)}\cos\theta + b_n^{(2)}\cos 2\theta + \cdots$$

where $n = 1, 3, 5, 7$.* In order to obtain the required derivatives Newcomb introduced a general set of quantities $c_n^{(i,j)}$, $j = 1, 2, \cdots, 6$, $c_n^{(i,0)} = c_n^{(i)}$. The quantity $D^m c_n^{(i)}$ is a linear function, with positive integral coefficients, of $c_n^{(i)}$, $c_n^{(i,1)}, \cdots, c_n^{(i,m)}$. The computation problem therefore reduces to the determination of the $c_n^{(i,j)}$. Newcomb's method of obtaining these numbers has been commended for its accuracy. It is, however, long, although addition and subtraction logarithms were employed to compute the continued fractions that are the foundation of the method. Apparently the same process had to be carried out for each of the values of n.† Newcomb's failure to discover certain remarkable recursion relations that allow the $c_3^{(i,j)}$ to be obtained from the $c_1^{(i,j)}$, the $c_5^{(i,j)}$ from the $c_3^{(i,j)}$, etc., both rapidly and accurately, probably came from his method of handling the quantities. He introduced them as hypergeometric functions with rather complicated parameters. He then apparently looked through the list of relations given by Gauss to find formulas that might be useful. Such a method will evidently fail to reveal relations that owe their origin to the special parameters involved.

2. In this paper a rapid and accurate method of computing the coefficients used by Newcomb will be developed. It is adapted throughout to machine computation. The recursion relations that are used will be obtained by contour integration. This seems the modern way of dealing with the

* For simplicity of typing the manuscript the letter s was used in place of the customary a, that is, $s = a/a'$, where a is the semi-major axis of the orbit of the inner planet and a' that of the outer.

† Newcomb's exposition is given in *Astronomical Papers of the American Ephemeris*, Vol. III, Chapters II and III. Tables of the constants for certain pairs of planets will be found in the last memoir of that volume, and in Vol. V of the same papers, pp. 339-348.

problem. By breaking up the integrand in various ways, integrating by parts, etc., one can hope to discover all important relations connecting the numbers being considered. Some of the relations employed are already in the literature, but they will be developed in order to give completeness to the method. For the same reason the relation that is used to express $D^m c_n^{(i)}$ in terms of the $c_n^{(i,j)}$ will be obtained by integration, although it is not required in the actual problem. Those of the $c_1^{(i,j)}$ that are calculated directly are found by series identical with or similar to series recently given by Brown.

I. THE FUNDAMENTAL EXPRESSIONS.

3. Let

$$G = 1 + s^2 - 2s\cos\theta, \tag{1}$$

and put

$$z = e^{\iota\theta}, \qquad \iota = (-1)^{1/2}$$

so that

$$G = 1 + s^2 - s(z + 1/z) \tag{2^i}$$

$$= (1/z - s)(z - s) \tag{2^{ii}}$$

$$= (z - s)(1 - sz)/z \tag{2^{iii}}$$

$$= -s(z - 1/s)(z - s)/z. \tag{2^{iv}}$$

We have from (2^{iv})

$$G^{-n/2} = \frac{1}{(-s)^{n/2}}\left[\frac{z}{(z - 1/s)(z - s)}\right]^{n/2}.$$

It is seen that $G^{-n/2}$ has branch points at $z = 0$, s, and $1/s$, the latter two points being also infinities. It is also seen that the function is single valued in the ring between $|z| = s$ and $|z| = 1/s$, so that within this region it can be expanded in a Laurent series. Therefore

$$G^{-n/2} = (1/2)\sum_{i=-\infty}^{\infty} b_n^{(i)} z^i.$$

Since

$$G^{-n/2}(1/z) = G^{-n/2}(z),$$

it follows that $b_n^{(-i)} = b_n^{(i)}$. We can accordingly write

$$G^{-n/2} = (1/2)\sum_{i=0}^{\infty} b_n^{(i)} z^i + (1/2)\sum_{i=1}^{\infty} b_n^{(i)} z^{-i}.$$

The expansion being valid on the unit circle we have

$$\begin{aligned}(1 + s^2 - 2s\cos\theta)^{-n/2} &= (1/2)\sum_{i=0}^{\infty} b_n^{(i)} e^{i\iota\theta} + (1/2)\sum_{i=1}^{\infty} b_n^{(i)} e^{-i\iota\theta} \\ &= (1/2) b_n^{(0)} + \sum_{i=1}^{\infty} b_n^{(i)} \cos i\theta. \end{aligned} \tag{3}$$

In accordance with the theorem of Laurent

$$b_n^{(i)} = (1/\pi\iota) \int z^{-i-1} G^{-n/2} dz, \tag{4}$$

where the integral may be taken around the unit circle. Since $b_n^{(-i)} = b_n^{(i)}$ we have

$$b_n^{(i)} = (1/\pi\iota) \int z^{i-1} G^{-n/2} dz. \tag{4'}$$

4. In order to obtain ultimately the $c_n^{(i,j)}$ of Newcomb we introduce the general set of quantities

$$g(i, j, n) = \int z^{i-1} (z - s)^j G^{-j-n/2} dz, \tag{5}$$

the path being the circle $| z | = 1$. The dependence of g upon s need not be exhibited.

The function g is expressible in terms of the hypergeometric function, and the relation between the two will be obtained for the case $n = 1$.

We have, using $(2''')$,

$$g(i, j, 1) = \int z^{j+i-1/2} (z - s)^{-1/2} (1 - sz)^{-j-1/2} dz,$$

or, putting $z = st$,

$$g(i, j, 1) = s^{j+1} \int t^{j+i-1/2} (t - 1)^{-1/2} (1 - s^2 t)^{-j-1/2} dt,$$

the path being the circle $t = 1/s$. The path can be changed into:

(1) The real axis from $t = 1/s$ to $t = 1 + \epsilon$,
(2) A semi-circle of radius ϵ about $t = 1$, above the real axis,
(3) The real axis from $t = 1 - \epsilon$ to $t = \epsilon$,
(4) A circle about $t = 0$,
(5) A line from $t = \epsilon$ to $t = 1 - \epsilon$,
(6) A semi-circle of radius ϵ about $t = 1$ and below the real axis.
(7) The real axis from $t = 1 + \epsilon$ to $t = 1/s$.

It is easily shown that the portions of the integral from (2), (4), and (6) approach 0 as $\epsilon \to 0$, that the portions from (1) and (7) cancel each other, and that those from (3) and (5) are equal. Therefore, since the description of (2) leads us to replace $(t - 1)^{-1/2}$ by

$$e^{-\pi\iota/2} (1 - t)^{-1/2} = -\iota (1 - t)^{-1/2}$$

we have

$$g(i, j, 1) = 2\iota s^{j+1} \int_0^1 t^{j+i-1/2} (1 - t)^{-1/2} (1 - s^2 t)^{-j-1/2} dt.$$

But it is well known that

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-uz)^{-a}du = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a,b,c,z)$$

where Γ is the ordinary gamma function and F is the hypergeometric function.*

It follows that

$$g(i,j,1) = 2\iota s^{j+i} \frac{\Gamma(j+i+1/2)\Gamma(1/2)}{\Gamma(j+i+1)} \times F(j+1/2, j+i+1/2, j+i+1, s^2).$$

Using the relations

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma^2(1/2) = \pi, \quad \Gamma(j+i+1) = (j+1)!,$$

we find

$$g(i,j,1) = \frac{\pi\iota s^{j+i}[1\cdot 3\cdots(2j+2i-1)]}{2^{j+i-1}(j+i)!} \times F(j+1/2, j+i+1/2, j+i+1, s^2). \tag{6}$$

5. Let us now put

$$c_n^{(i,j)} = L(j,n)g(i,j,n), \tag{7}$$

where

$$L(0,n) = (1/\pi\iota)s^{(n-1)/2},$$
$$L(j,n) = (1/\pi\iota)n(n+2)\cdots(n+2j-2)s^{(n+2j-1)/2}. \tag{8}$$

For simplicity we set

$$c_n^{(i)} = c_n^{(i,0)}. \tag{9}$$

We have

$$c_n^{(i)} = (1/\pi\iota)s^{(n-1)/2}\int z^{i-1}G^{-n/2}dz,$$

so that

$$c_n^{(i)} = s^{(n-1)/2}b_n^{(i)}. \tag{10}$$

In particular,

$$c_1^{(i)} = b_1^{(i)}. \tag{11}$$

II. THE COMPUTATION OF SERIES FOR THE $c_1^{(i,j)}$.

6. The present method requires the direct calculation of certain of the $c_1^{(i,j)}$. Brown has given rapidly, converging series for $c_1^{(10)}$ and $c_1^{(11)}$†; these with a well known recursion relation will give the remaining $c_1^{(i)}$. We need

* Whittaker and Watson, *Modern Analysis*, 3rd Ed., p. 293.

† *Monthly Notices of the Royal Astronomical Society*, Vol. 88, pp. 459-469. As a matter of fact we use a series for $c_1^{(9)}$ in place of Brown's series for $c_1^{(11)}$.

also the values of $c_1^{(10,j)}$, $j = 1, 2, \cdots, 6$. Series similar to those of Brown will be obtained for $j = 1, 2, 3$; recursion relations will be used for $j = 4, 5, 6$.

7. From (6) and (8) we have

(12)
$$c_1^{(i,j)} = \frac{[1\cdot 3\cdots(2j-1)]\,[1\cdot 3\cdots(2j+2i-1)]}{2^{j+i-1}(j+i)!}\, s^{2j+i}\, F(a,b,c,s^2),$$

where

(13) $$a = j + 1/2, \quad b = j + i + 1/2, \quad c = j + i + 1.$$

It follows from the differential equation for the hypergeometric function that $F(a, b, c, s^2)$ satisfies the equation

(14) $$s(s^2-1)(d^2F/ds^2) + [(2a+2b+1)s^2 - 1 - 2c](dF/ds) + 4absF = 0.$$

Let us now make the change of variables

(15) $$s^2 = y/(1+y).$$

We then have from (12)

(16)
$$c_1^{(i,j)} = \frac{2^j[1\cdot 3\cdots(2j-1)]\,[1\cdot 3\cdots(2j+2i-1)]}{(j+i)!}\left(\frac{s}{2}\right)^{2j+i-1} y^{1/2}H,$$

where

(17) $$H(a,b,c,y) = (1+y)^{-1/2}\, F\{a, b, c, y/(1+y)\}.$$

Using (14) it is found that H satisfies the equation

$$4y(1+y)^3(d^2H/dy^2) + 4(1+y)^2[c + (2+c-a-b)y](dH/dy) + (1+y)[2c - 4ab + (1+2c-2a-2b)y]H = 0.$$

If we make use of (13) and denote H when expressed in terms of i, j, and y by $H(i, j, y)$ we find

(18) $$4y(1+y)^2H''(i,j,y) + 4(1+y)[i+j+1+(2-j)y]H'(i,j,y) + [1-2j-4ij-4j^2+(1-2j)y]H(i,j,y) = 0.$$

8. The equation (18) has the one singular point $y = -1$, but is regular at that point. It appears from (17) that the value of H desired is the expansion about $y = 0$ in which the constant term has the value 1. The equation (18) is satisfied by the series

(19) $$1 + a_1y + a_2y^2 + \cdots$$

where, in particular,

$$a_1 = (2j + 4ij + 4j^2 - 1)/4(i+j+1).$$

The value of y in terms of s is

$$y = s^2/(1-s^2). \tag{20}$$

Thus y is positive since $0 < s < 1$. But $y > 1$ if $s > 2^{\frac{1}{2}}/2$. Since the series (19) does not converge for $y > 1$, on account of the singular point $y = -1$, it is necessary, in certain cases, to use the analytic continuation of the series. This is easily done; the following considerations will govern the development.

The greatest value of s for the major planets is for Venus-Earth, for which $s = .723$, giving $y = 1.097$. We will derive an expansion for H about $y = .6$. Such an expansion will converge for $|y - .6| < 1.6$, since the only singular point of (16) is $y = -1$.

Putting $t = y - .6$, we write (18) in the form

$$\begin{aligned} &4(.6+t)(1.6+t)^2H''(i,j,t) \\ &+4(1.6+t)[i+j+1+.6(2-j)+(2-j)t]H'(i,j,t) \\ &+[1-2j-4ij-4j^2+.6(1-2j)+(1-2j)t]H(i,j,t)=0. \end{aligned} \tag{21}$$

This equation is satisfied by

$$H = b_0 + b_1 t + b_2 t^2 + \cdots,$$

where b_0 is obtained by substituting $y = .6$ in (19) and b_1 by substituting $y = .6$ in

$$a_1 + 2a_2 y + 3a_3 y^2 + \cdots.$$

The other coefficients are determined in terms of b_0 and b_1.

The actual series for the values of i and j required are given in V. It will be seen that they converge rapidly.

III. The Recursion Relations.

9. In order to obtain the recursion relations for the $c_n^{(i,j)}$ we establish such relations for the functions $g(i, j, n)$.

We begin by deducing by contour integration a well known relation connecting the values of $c_1^{(i)}$ for three successive values of i.

We have from (5)

$$g(i,0,1) = \int z^{i-1} G^{-n/2} dz.$$

Observing from (2^i) that

$$\partial G/\partial z = -s + s/z^2,$$

we see that we can write

$$g(i,0,1) = -(1/s)\int z^{i-1} G^{-n/2}(\partial G/\partial z)\,dz + g(i-2,0,1).$$

If we integrate by parts, observing that on account of the single valued property of $G^{-n/2}$ along the path of integration, the integrated term in the familiar formula is zero, we obtain readily

$$g(i,0,1)=\frac{2i-2}{2i-1}\,(s+1/s)g(i-1,0,1)-\frac{2i-3}{2i-1}\,g(i-2,0,1).$$

From (7) and (8) it follows that

(22)
$$c_1^{(i)}=\frac{2i-2}{2i-1}(s+1/s)c_1^{(i-1)}-\frac{2i-3}{2i-1}c_1^{(i-2)}.$$

10. We have

$$g(i,j+2,n-2)=\int z^{i-1}(z-s)^{j+2}G^{-j-1-n/2}dz$$
$$=\int z^{i}(z-s)^{j+1}G^{-j-1-n/2}dz-s\int z^{i-1}(z-s)^{j+1}G^{-j-1-n/2}dz,$$

from which we obtain

(23) $$g(i+1,j+1,n)=sg(i,j+1,n)+g(i,j+2,n-2).$$

If we multiply the two members of this relation by $L(j+2,n-2)$, and observe that

(24) $$L(j+2,n-2)=(n-2)L(j+1,n),$$

we find

$$c_n^{(i+1,j+1)}=sc_n^{(i,j+1)}+\frac{1}{n-2}\,c_{n-2}^{(i,j+2)},$$

or, changing i into $i-1$, and j into $j-1$,

(25) $$c_n^{(i,j)}=sc_n^{(i-1,j)}+\frac{1}{n-2}c_{n-2}^{(i-1,j+1)}.$$

11. From (2^{ii}) it follows that

(26) $$G^{-j-n/2}=G^{-j-1-n/2}(1/z-s)(z-s),$$

from which it is seen that

(27) $$g(i,j,n)=g(i-1,j+1,n)-sg(i,j+1,n).$$

Multiplying by $L(j+1,n)$ and observing that

$$L(j+1,n)=(n+2j)sL(j,n),$$

we obtain

(28) $$c_n^{(i-1,j+1)}=s[c_n^{(i,j+1)}+(n+2j)c_n^{(i,j)}].$$

12. Replacing i by $i-1$ in (27) we have

$$(29)\qquad g(i-1,j,n)=g(i-2,j+1,n)-sg(i-1,j+1,n).$$

On the other hand we have by integrating by parts, putting

$$dv=(z-s)^j dz,$$

the result

$$g(i,j,n)=-\frac{i-1}{j+1}\int (z-s)^{j+1}z^{i-2}G^{-j-n/2}dz$$
$$-\frac{s(n+2j)}{2(j+1)}\int (z-s)^{j+1}z^{i-1}(1-1/z^2)\,G^{-j-1-n/2}dz$$

$$=-\frac{i-1}{j+1}\int z^{i-1}(z-s)^j G^{-j-n/2}dz+\frac{i-1}{j+1}\,s\int z^{i-2}\,(z-s)^j G^{-j-n/2}dz$$
$$-\frac{s(n+2j)}{2(j+1)}\int z^{i-1}(z-s)^{j+1}G^{-j-1-n/2}dz$$
$$+\frac{s(n+2j)}{2(j+1)}\int z^{i-3}(z-s)^{j+1}G^{-j-1-n/2}dz$$

$$=-\frac{i-1}{j+1}\,g(i,j,n)+\frac{(i-1)s}{j+1}\;g(i-1,j,n)$$
$$-\frac{s(n+2j)}{2(j+1)}\;g(i,j+1,n)+\frac{s(n+2j)}{2(j+1)}\;g(i-2,j+1,n).$$

From this we derive

$$(30)\qquad 2(j+1)g(i,j,n)-2(i-1)sg(i-1,j,n)$$
$$=(n+2j)sg(i-2,j+1,n)-(n+2j)sg(i,j+1,n).$$

Eliminating $g(i-2,j+1,n)$ between (29) and (30) and then eliminating $g(i-1,j+1,n)$ by means of (27), we have

$$(31)\qquad (n+2j+2i-2)sg(i-1,j,n)-[2(j+i)-(n+2j)s^2]g(i,j,n)$$
$$=(n+2j)s(1-s^2)g(i,j+1,n).$$

Multiplying (31) by $L(j+1,n)$ and afterwards replacing j by $j+1$, we have

$$(32)\qquad (n+2j+2i)sc_n^{(i-1,j+1)}-[2(j+i+1)-(n+2j+2)s^2]c_n^{(i,j+1)}$$
$$=(1-s^2)c_n^{(i,j+2)}.$$

12. In order to obtain the next relation it is convenient to set

$$A=g(i+1,j,n),\qquad X=g(i-1,j+1,n),$$
$$B=g(i,j+1,n-2),\qquad Y=g(i,j+1,n),$$
$$C=g(i,j+2,n-2),\qquad Z=g(i+1,n+1,n).$$

Using (26) it is seen that

$$A = Y - sZ. \tag{33}$$

On the other hand

$$C = -sY + Z. \tag{34}$$

If we integrate by parts we obtain

$$B = -\frac{j+1}{i} A - \frac{s(n+2j)}{2i} Z + \frac{s(n+2j)}{2i} X,$$

so that

$$2iB + 2(j+1)A = s(n+2j)(X - Z). \tag{35}$$

But if we put

$$G^{-j-n/2} = (1 + s^2 - sz - s/z) G^{-j-1-n/2},$$

we find

$$B = (1+s^2)Y - sZ - sX. \tag{36}$$

Eliminating X between (35) and (36) we obtain

$$(n+2j+2i)B + 2(j+1)A = (n+2j)[(1+s^2)Y - 2sZ].$$

But from (33) and (34)

$$A - sC = (1+s^2)Y - 2sZ,$$

so that

$$(n+2j+2i)B + 2(j+1)A = (n+2j)(A - sC).$$

We thus obtain finally

$$(n-2)g(i+1, j, n) = (n+2j+2i)g(i, j+1, n-2) + s(n+2j)g(i, j+2, n-2). \tag{37}$$

If we multiply by $L(j+2, n-2)$ and use (24) as well as the relation

$$L(j+2, n-2) = (n+2j)sL(j+1, n-2)$$

we find

$$(n-2)^2 c_n^{(i+1,j)} = (n+2j+2i)\, c_{n-2}^{(i,j+1)} + c_{n-2}^{(i,j+2)}, \tag{38}$$

a relation obtained by Innes in a different way.*

If we eliminate $c_n^{(i-1,j+1)}$ between (28) and (32) we find

$$c_n^{(i,j+2)} = (n+2j)(n+2j+2i)[s^2/(1-s^2)]c_n^{(i,j)} - 2\{(n+2j+i+1) - (n+j)[1/(1-s^2)]\}c_n^{(i,j+1)}. \tag{39}$$

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* *Monthly Notices of the Royal Astronomical Society*, Vol. 69, p. 639. The relation is however printed erroneously.

All the necessary recursion relations have now been obtained.

13. Although the aim of the present paper is to give a convenient method for computing the $c_n^{(i,j)}$, we shall for the sake of completeness, as stated in § 2, obtain the basic formula by which the $D^m c_n^{(i)}$ are expressed in terms of the $c_n^{(i,j)}$.

We have

$$s(\partial g/\partial s) = -js\int z^{i-1}(z-s)^{j-1}G^{-j-n/2}dz$$
$$-(j+n/2)s\int z^{i-1}(z-s)^{j}G^{-j-1-n/2}(\partial G/\partial s)dz.$$

But

$$\partial G/\partial s = -2(z-s)-(z/s)\partial G/\partial z.$$

If we use this expression and integrate by parts we find

$$s(\partial g/\partial s) = (i+j)g(i,j,n)+(n+2j)sg(i,j+1,n). \tag{40}$$

Put

$$D = s(\partial/\partial s);$$

then from (7) we have

$$Dc_n^{(i,j)} = g(i,j,n)DL(j,n)+L(j,n)Dg(i,j,n). \tag{41}$$

If now we multiply (40) by $L(j,n)$ and use the relation

$$(n+2j)sL(j,n) = L(j+1,n)$$

to transform it we find by (41), since

$$DL(j,n) = [(n+2j-1)/2]\,L(j,n),$$

the ultimate relation

$$Dc_n^{(i,j)} = [(n+2i+4j-1)/2]\,c_n^{(i,j)} + c_n^{(i,j+1)}. \tag{42}$$

We have for instance

$$Dc_n^{(i)} = [(n+2i-1)/2]\,c_n^{(i)} + c_n^{(i,1)}$$
$$D^2c_n^{(i)} = [(n+2i-1)/2]^2c_n^{(i)} + (n+2i+1)c_n^{(i,1)} + c_n^{(i+2)},$$

etc.*

* It is not desirable to give general algebraic expressions for $Dmc_n^{(i)}$ for larger values of m, but merely to compute the numerical coefficients. A table of the coefficients is given by Newcomb, *loc. cit.*, V., p. 313.

IV. OUTLINE OF THE COMPUTATIONS.

14. To compute the $c_1^{(i,j)}$.*

The plan is to compute each column of the array starting from the bottom.

(1) From the value of s compute the values of

$$(43)\qquad \begin{aligned} y &= s^2/(1-s^2), & s_1 &= 1/(1-s^2),\\ s' &= s+1/s, & s_2 &= s/(1-s^2). \end{aligned}$$

(2) From the series expansions compute the values of $c_1^{(9)}$ and $c_1^{(10)}$.

(3) From (22) we have

$$(44)\qquad c_1^{(i-2)} = \frac{2i-2}{2i-3}s'c_1^{(i-1)} - \frac{2i-1}{2i-3}c_1^{(i)}.$$

Beginning with the values of $c_1^{(10)}$ and $c_1^{(9)}$ compute successively $c_1^{(8)}, \cdots$, and $c_1^{(0)}$.†

(4) Using the series expansions compute $c_1^{(10,1)}$, $c_1^{(10,2)}$, $c_1^{(10,3)}$.

(5) From (28) we have, for $n=1$,

$$(45)\qquad c_1^{(i-1,j+1)} = s(c_1^{(i,j+1)} + (1+2j)c_1^{(i,j)}).$$

This formula enables us to complete the second, third, and fourth columns, by working from the bottom upwards.‡ The computation is made to include the case $i=0$, so as to give $c_1^{(-1,1)}$, $c_1^{(-1,2)}$, $c_1^{(-1,3)}$. Such quantities were not employed by Newcomb, but are needed here in order to start the table of the $c_3^{(i,j)}$.

(6) The remaining columns are found in a similar way except that the bottom elements are gotten by recursion relations instead of from series. From (32) we find, on putting $n=1$ and $j=2$, 3, 4, successively,

$$(46)\qquad \begin{aligned} c_1^{(10,4)} &= 25s_2c_1^{(9,3)} - (26s_1 - 7y)c_1^{(10,3)},\\ c_1^{(10,5)} &= 27s_2c_1^{(9,4)} - (28s_1 - 9y)c_1^{(10,4)},\\ c_1^{(10,6)} &= 29s_2c_1^{(9,5)} - (30s_1 - 11y)c_1^{(10,5)}. \end{aligned}$$

(7) We also need $c_1^{(-1,7)}$. From (39) we have, putting $n=1$, $i=-1$, $j=5$,

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* It is assumed that the $c_1^{(i,j)}$ are needed up to $i=10$, $j=6$. More rows and columns can be added if necessary. For small values of s the entire computation can be shortened by methods that will suggest themselves.

† Since $c_1^{(0)}$ is an elliptic integral of the first kind, it can be computed by a rapidly converging theta series, thus checking the first column.

‡ If we solve (45) for $c_1^{(i,j+1)}$ we have a means, in conjunction with (44), to work downwards, adding new rows.

$$(47) \qquad c_1^{(-1,7)} = 99yc_1^{(-1,5)} - (22 - 12s_1)c_1^{(-1,6)}.$$

15. To calculate the $c_3^{(i,j)}$.

(1) Putting $n = 3$, $i = -1$ in (38) we have

$$(48) \qquad c_3^{(0,j)} = (1 + 2j)c_1^{(-1,j+1)} + c_1^{(-1,j+2)},$$

which gives the first row in the table up to $j = 5$.

(2) Putting $n = 3$ in (25) we have

$$(49) \qquad c_3^{(i,j)} = sc_3^{(i-1,j)} + c_1^{(i-1,j+1)},$$

which allows the table to be extended to as great a value of i as desired.

(3) We need the values of $c_3^{(-1,j)}$ in order to start the table of the $c_5^{(i,j)}$. From (28) we have

$$(50) \qquad c_3^{(-1,j+1)} = s[c_3^{(0,j+1)} + (3 + 2j)c_3^{(0,j)}].$$

This will give the values from $c_3^{(-1,1)}$ to $c_3^{(-1,5)}$. We also require $c_3^{(-1,6)}$. From (39)

$$(51) \qquad c_3^{(-1,6)} = 99yc_3^{(-1,4)} - (22 - 14s_1)c_3^{(-1,5)}.$$

16. The calculation of the $c_5^{(i,j)}$ is similar to that of the $c_3^{(i,j)}$, etc.

It is seen that there will be one fewer columns in the array of the $c_3^{(i,j)}$ than in that of the $c_1^{(i,j)}$. If necessary, a new column can be added to the $c_3^{(i,j)}$ as the last columns were added in the case of the $c_1^{(i,j)}$, calculating the final element of the column by (32) and working up the column by (28).

The recursion formulas that form the distinctive feature of the present method are (25) and (28). They are the only ones that are repeatedly used with the exception of the classic formula (22). These formulas possess in a marked degree the characteristics essential for computational purposes. They do not lead to an accumulation of errors for coefficients are small and there are no negative terms, and furthermore they are rapidly used. Formula (25) is particularly accurate for $n = 5, 7$. The coefficients in (25) do not depend upon either i or j, while those in (28) are constant down a column. This is a great advantage. Formula (32) does not have the merit of (25) and (28). This is seen by inspecting (46). Errors accumulate through its repeated use. On this account it is desirable to compute $c_1^{(10,3)}$ to two more places that will be retained in $c_1^{(10,4)}$ etc. The rapidity of convergence of the series makes this practical. A similar criticism can be made of (39). No extensive use of it is made, however.* For smaller values of i, it may happen

* Newcomb's method of obtaining approximate values for numbers in columns to the right of those accurately computed was solely one of induction.

that (39) gives $c_n^{(i,j)}$ as the sum and not the difference, of two quantities, and the formula becomes more accurate.*

V. THE NUMERICAL SERIES.

17. The series employed in § 14 (2) and (4), and whose derivation was explained in II follow.† The series in y are used if $y \leqq .3$, otherwise the series in $t = y - .6$.

$c_1^{(9)} = 4 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 (s/4)^8 y^{\frac{1}{2}}$ multiplied by one of the series

1.000 000 00	0.985 836 16
$-$ 0.025 000 00y	$-$ 0.022 336 71t
$+$ 0.002 556 82y^2	$+$ 0.001 933 39t^2
$-$ 0.000 443 89y^3	$-$ 0.000 270 37t^3
$+$ 0.000 104 57y^4	$+$ 0.000 049 43t^4
$-$ 0.000 030 25y^5	$-$ 0.000 010 81t^5
$+$ 0.000 010 17y^6	$+$ 0.000 002 75t^6
$-$ 0.000 003 84y^7	
$+$ 0.000 001 59y^8	
$-$ 0.000 000 71y^9	

$c_1^{(10)} = 2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 (s/4)^9 y^{\frac{1}{2}}$ multiplied by one of the series

1.000 000 00	0.987 065 29
$-$ 0.022 727 27y	$-$ 0.020 485 38t
$+$ 0.002 130 68y^2	$+$ 0.001 643 37t^2
$-$ 0.000 341 45y^3	$-$ 0.000 214 74t^3
$+$ 0.000 074 69y^4	$+$ 0.000 036 71t^4
$-$ 0.000 020 17y^5	$-$ 0.000 006 78t^5
$+$ 0.000 006 35y^6	
$-$ 0.000 002 26y^7	
$+$ 0.000 000 88y^8	

$c_1^{(10,1)} = 16 \cdot 13 \cdot 17 \cdot 19 \cdot 21 (s/4)^{11} y^{\frac{1}{2}}$ multiplied by one of the series

1.000 000 00	1.544 629 77
$+$ 0.937 500 00y	$+$ 0.880 010 95t
$-$ 0.053 485 58y^2	$-$ 0.043 047 37t^2
$+$ 0.007 136 42y^3	$+$ 0.004 720 19t^3
$-$ 0.001 385 03y^4	$-$ 0.000 726 91t^4
$+$ 0.000 340 46y^5	$+$ 0.000 137 82t^5
$-$ 0.000 099 06y^6	$-$ 0.000 030 24t^6
$+$ 0.000 032 77y^7	$+$ 0.000 007 50t^7
$-$ 0.000 012 01y^8	$-$ 0.000 002 34t^8
$+$ 0.000 004 78y^9	
$-$ 0.000 002 04y^{10}	
$+$ 0.000 000 93y^{11}	

* For instance, in the case of Venus-Earth, if we determine $c_1^{(1,6)}$ from (39) using Newcomb's values for $c_1^{(1,4)}$ and $c_1^{(1,5)}$ we find precisely his value to all seven places.

† The series in y for $c_1^{(10)}$ is the same as one of the series in Brown's paper except as to the number of terms and decimal places. Brown also gives expansions about $y = .5$ and $y = 1$. All the series here given were computed by Miss Irene Price.

$c_1^{(10,2)} = 32 \cdot 13 \cdot 17 \cdot 19 \cdot 21 \cdot 23 (s/4)^{13} y^{\frac{1}{2}}$ multiplied by one of the series

1.000 000 00	2.426 804 77
$+ 1.903\ 846\ 15y$	$+ 2.832\ 665\ 16t$
$+ 0.825\ 721\ 15y^2$	$+ 0.727\ 879\ 22t^2$
$- 0.064\ 603\ 37y^3$	$- 0.045\ 893\ 56t^3$
$+ 0.010\ 370\ 11y^4$	$+ 0.005\ 842\ 58t^4$
$- 0.002\ 262\ 36y^5$	$- 0.000\ 987\ 77t^5$
$+ 0.000\ 600\ 84y^6$	$+ 0.000\ 199\ 02t^6$
$- 0.000\ 184\ 17y^7$	$- 0.000\ 045\ 05t^7$
$+ 0.000\ 063\ 11y^8$	$+ 0.000\ 009\ 92t^8$
$- 0.000\ 023\ 66y^9$	
$+ 0.000\ 009\ 56y^{10}$	
$- 0.000\ 004\ 12y^{11}$	
$+ 0.000\ 001\ 87y^{12}$	
$- 0.000\ 000\ 89y^{13}$	

$c_1^{(10,3)} = 256 \cdot 17 \cdot 19 \cdot 21 \cdot 23 \cdot 25 (s/4)^{15} y^{\frac{1}{2}}$ multiplied by one of the series *

1.000 000 00	3.824 315 72
$+ 2.875\ 000\ 00y$	$+ 6.768\ 654\ 73t$
$+ 2.653\ 125\ 00y^2$	$+ 3.801\ 102\ 49t^2$
$+ 0.701\ 318\ 36y^3$	$+ 0.583\ 328\ 93t^3$
$- 0.062\ 099\ 50y^4$	$- 0.039\ 025\ 67t^4$
$+ 0.011\ 030\ 56y^5$	$+ 0.005\ 305\ 42t^5$
$- 0.002\ 588\ 22y^6$	$- 0.000\ 941\ 87t^6$
$+ 0.000\ 723\ 77y^7$	$+ 0.000\ 193\ 22t^7$
$- 0.000\ 230\ 04y^8$	$- 0.000\ 033\ 27t^8$
$+ 0.000\ 080\ 82y^9$	
$- 0.000\ 030\ 81y^{10}$	
$+ 0.000\ 012\ 58y^{11}$	
$- 0.000\ 005\ 44y^{12}$	
$+ 0.000\ 002\ 48y^{13}$	
$- 0.000\ 001\ 18y^{14}$	
$+ 0.000\ 000\ 58y^{15}$	

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* In spite of the number of integers that occur in the coefficients of the various series the power of $s/4$ reduces it in every case to a small quantity. For example in the case of Venus-Earth, for which $\log s = 9.85\ 9337$, $\log y = 0.04\ 0346$, the coefficient of the series is found to be 0.00 03601.

A System of Six Rectangular Biquadratics.

By Mildred Waters Dean.

This paper discusses a special set of six points in the inversive plane, and certain biquadratic curves related to them. It takes up the following topics:

I. Representing the three vertices of a triangle by complex quantities a, b, and c, and studying the locus traced out by a point moving so as to subtend equal angles (modulo π) at two sides ab and ac of the triangle, we know that this locus is a biquadratic having a double point at a, cutting itself at right angles there, and passing through the other two vertices of the triangle, its two Fermat points (denoted by f and f_1) and the point ∞. Then, in addition to the biquadratic cutting itself at right angles at a, and passing through b, c, f, f_1 and ∞, there are five other curves which bear the same relation to the set of six points a, b, c, f, f_1 and ∞, namely: five biquadratics cutting themselves at right angles at b, c, f, f_1 and ∞, respectively, and passing through the other five points of the set. Thus, the three vertices of the triangle, the Fermat points, and the point ∞ are a special set of six points such that there can be found six biquadratic curves, each having a double point at one of the six points, cutting itself at right angles there, and passing through the other five.

Moreover, it is shown that this set of six points can be broken up into two triads which are apolar.

II. The study of the conditions under which, given a set of six points, the biquadratic having a double point at each and passing through the others will be rectangular. (A rectangular curve is one cutting itself at right angles at a double point.) A set of six points in the inversive plane is chosen, with the restriction that one of them be the point ∞, and the necessary conditions are derived.

III. We next give the geometrical interpretation of the conditions derived in Section II. It is known that five points have a biquadratic invariant I_2.* For any given four points this is a covariant biquadratic curve C_2. The relation of the four and any point on C_2 is symmetrical. It is found that the set of six points mentioned in Section II is such that I_2 vanishes for any five, or, in other words, that the covariant C_2 of any four is on the other two.

* This invariant I_2 is given in a "Note on Neuberg's Cubic Curve," by Professor Morley, *American Mathematical Monthly*, Vol. 32 (1925), pp. 407-411.

IV. It is then pointed out that the six points, together with the associated biquadratics, may be taken as the fundamental points and curves of a Geiser involution in the inversive plane.

I.

We have before us the problem of finding the locus of a point which moves so as to subtend equal angles, modulo π, at two sides of a given triangle *abc.** (By "equal modulo π" we mean that either the angles are equal, or they differ by π.) If we denote our moving point by a complex quantity z, we may first try to find its locus when it moves so as to subtend equal angles (modulo π) at the sides *ab* and *ac*. The line from b to z is a vector $z - b = \rho_1 e^{i\theta_1}$. Likewise, the line from a to z is a vector $z - a = \rho_2 e^{i\theta_2}$ and that from c to z is a vector $z - c = \rho_3 e^{i\theta_3}$. Now we wish the angles *azb* and *cza* to be equal, or else to differ by π. If we take the quotient $(z-b)/(z-a)$, we find that it is equal to $(\rho_1/\rho_2)e^{i(\theta_1-\theta_2)}$, where $\theta_1 - \theta_2$ is the angle *azb*. Similarly, $(z-a)/(z-c)$ is equal to $(\rho_2/\rho_3)e^{i(\theta_2-\theta_3)}$, where $\theta_2 - \theta_3$ is the angle *cza*. Now form the quotient

$$[(z-b)/(z-a)]/[(z-a)/(z-c)] = (\rho_1\rho_3/\rho_2^2)e^{i(\theta_1+\theta_3-2\theta_2)}.$$

If the angles *azb* and *cza* are to be equal, we must have $\theta_1 - \theta_2 = \theta_2 - \theta_3$, or $\theta_1 + \theta_3 - 2\theta_2 = 0$. If $azb - cza = \pi$, we have $\theta_1 - 2\theta_2 + \theta_3 = \pi$. Thus, the exponent of "e" in the quotient $[(z-b)/(z-a)]/[(z-a)/(z-c)]$ either vanishes or reduces to $i\pi$, so that, in either case, the quotient reduces to a real quantity. Therefore, we have as a first fundamental equation:

$$(1) \qquad [(z-b)/(z-a)]/[(z-a)/(z-c)] = \text{a real quantity.}$$

But, since a real quantity always equals its conjugate, we may write

$$(2) \qquad [(z-b)/(z-a)]/[(z-a)/(z-c)] = [(\bar{z}-\bar{b})/(\bar{z}-\bar{a})]/[(\bar{z}-\bar{a})/(\bar{z}-\bar{c})],$$

$$(3) \qquad (z-a)^2(\bar{z}-\bar{b})(\bar{z}-\bar{c}) - (\bar{z}-\bar{a})^2(z-b)(z-c) = 0.$$

* This is, of course, an old problem. See, for example, Salmon's "Higher Plane Curves." The problem is also discussed in an article by Professor M. T. Naraniengar, in the *Proceedings of the Edinburgh Mathematical Society*, Vol. 28, p. 73. The curve was called by the Belgian geometers the "focale à noeud." It is identical with the strophoid, which was first considered by Barrow in *Lectiones Geometricae* (1669), p. 69. Further references are: Van Rees, *Correspondance Mathématique de Quetelet*, tome V, p. 361; Lebon, *Journal de Mathématiques Speciales* (1895); Teixera, *Traité des Courbes Speciales Rémarkables*, tome I, p. 30. A history of the problem is given in Loria's *Speciale Algebraische und Transcendente Ebene Kurven* (1902), pp. 66-67, where various additional references are given.

The curve given by (3) is, inversively speaking, a biquadratic passing through ∞; projectively speaking, it is a circular cubic.* We shall think of it as a biquadratic. It is obvious that the curve passes through a, b, and c. Since the two Fermat points subtend equal angles, modulo π, at the sides of the given triangle, it is evident that our curve also passes through these points.

If we consider the point z in a position very near to a, we know, of course, by hypothesis, that angle azb = angle cza. The angles abz and acz have become infinitesimal, and may be considered as equal. Then the angle baz = angle zac; that is, the angles at the vertex a become more nearly equal as z approaches a. This shows that the curve (3), passing through a, bisects the angle at a. In fact, it can be shown that this curve not only bisects the angle at a, but it has a double point there, and cuts itself at right angles at this point. Let us denote this curve by A.

But A is not the only curve which furnishes us with a solution of our problem. For we may also have

$$(4) \qquad (z-c)^2(\bar{z}-\bar{a})(\bar{z}-\bar{b})-(\bar{z}-\bar{c})^2(z-a)(z-b)=0.$$

This is the locus of a point z which moves so as to subtend equal angles. mod. π, at the sides ac and bc. It is a biquadratic on ∞ which passes through c, cuts itself at right angles there, and bisects the angle at c. It also passes through a and b and the two Fermat points. We denote this curve by C. Similarly,

$$(5) \qquad (z-b)^2(\bar{z}-\bar{a})(\bar{z}-\bar{c})-(\bar{z}-\bar{b})^2(z-a)(z-c)=0$$

will be the locus of a point z moving so as to subtend equal angles, mod. π, at the sides bc and ba. It is also a biquadratic passing through a, b, c, f, f_1 and ∞. It cuts itself at right angles at b, and bisects the angle at b. We shall speak of this curve as the curve B.

So we have found three different curves which are solutions of our problem. We thus have a net of biquadratics on the six points a, b, c, f, f_1 and ∞. Now, since our biquadratics have double points at a, b, and c, respectively, and cut themselves at right angles at these points, is it not natural to suppose that we can find three more curves of the net, which have double points at f, f_1 and ∞, respectively, and which cut themselves at right angles at these points? This assumption is found to be true.

No simple method of obtaining the equations of the two biquadratics having double points at f and f_1 has been found, but we may find the equation of the one having a double point at ∞ as follows: the curve will have an

* See explanatory note at the end of this paper.

equation of the form $LA + MB + NC = 0$, where L, M, and N are real constants, which are to be determined. They must be determined in such a way that the curve will cut itself at right angles at ∞. This means that the coefficients of the terms in $z^2\bar{z}^2$, $z^2\bar{z}$, $z\bar{z}^2$ and $z\bar{z}$ must all be zero. The coefficient of $z^2\bar{z}^2$ is zero already. Equating the coefficient of $z^2\bar{z}$ to zero we have

$$(6) \qquad L(2\bar{a} - \bar{b} - \bar{c}) + M(2\bar{b} - \bar{a} - \bar{c}) + N(2\bar{c} - \bar{a} - \bar{b}) = 0.$$

Equating the coefficient of $z\bar{z}^2$ to zero we have

$$(7) \qquad L(b + c - 2a) + M(a + c - 2b) + N(a + b - 2c) = 0.$$

Equating the coefficient of $z\bar{z}$ to zero,

$$(8) \qquad L[2a(\bar{b} + \bar{c}) - 2\bar{a}(b + c)] + M[2b(\bar{a} + \bar{c}) - 2\bar{b}(a + c)] + N[2c(\bar{a} + \bar{b}) - 2\bar{c}(a + b)] = 0.$$

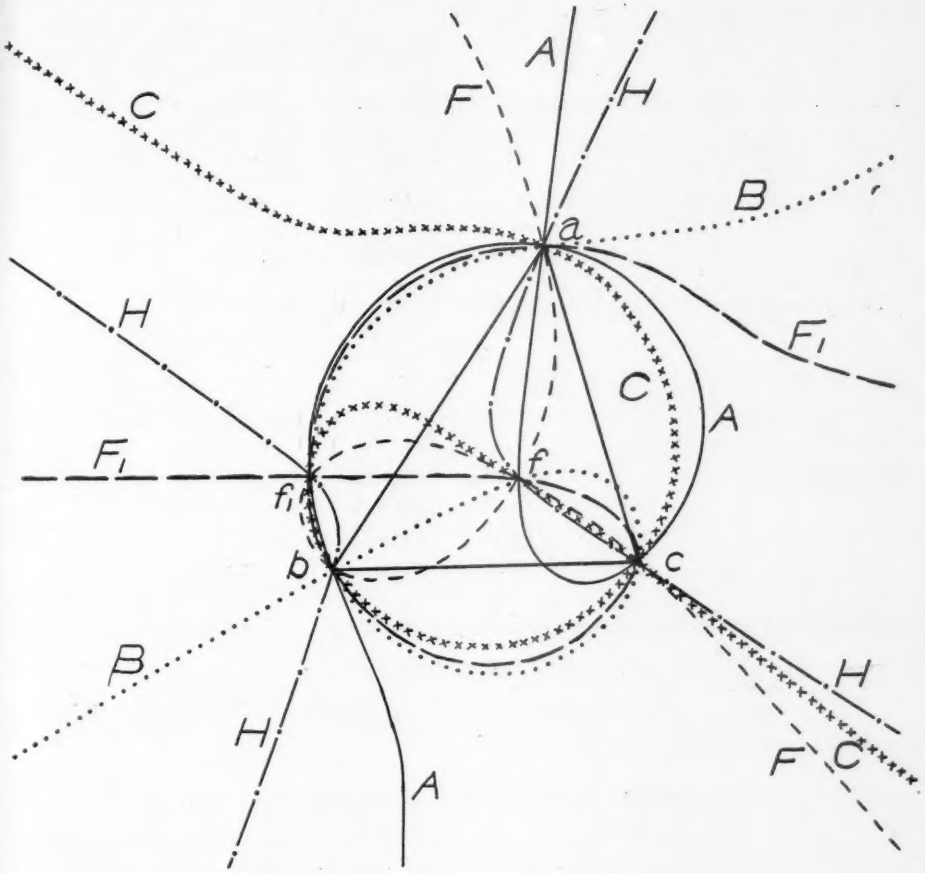
In (6), (7), and (8) we have three linear homogeneous equations in three unknowns L, M, and N. They have solutions different from zero only if

$$\begin{vmatrix} 2\bar{a} - \bar{b} - \bar{c} & 2\bar{b} - \bar{a} - \bar{c} & 2\bar{c} - \bar{a} - \bar{b} \\ b + c - 2a & a + c - 2b & a + b - 2c \\ 2a(\bar{b} + \bar{c}) - 2\bar{a}(b + c) & 2b(\bar{a} + \bar{c}) - 2\bar{b}(a + c) & 2c(\bar{a} + \bar{b}) - 2\bar{c}(a + b) \end{vmatrix} = 0$$

Adding the sum of the second and third columns to the first makes each element of the first column zero. Obviously, then, the values $L = M = N = 1$ will satisfy equations (6), (7), and (8), so that our biquadratic having a double point at infinity and cutting itself at right angles there is $A + B + C = 0$. A biquadratic cutting itself at right angles at infinity is known to be a rectangular hyperbola. In our case, the curve also passes through the three vertices of the triangle and its two Fermat points. The rectangular hyperbola passing through the three vertices of a triangle and its two Fermat points is known as the Kiepert hyperbola of the triangle. So our curve is nothing more than the Kiepert hyperbola of the triangle.

We may now state the following theorem:

THEOREM 1. *The three vertices of a triangle, its two Fermat points, and the point infinity form a set of six points such that the biquadratic having a double point at any one of them, and passing through the others, is rectangular.*



THE SIX RECTANGULAR BIQUADRATICS ASSOCIATED WITH THE TRIANGLE *abc*.

CURVE *A*—Biquadratic with Double Point at *a* ————————
CURVE *B*—Biquadratic with Double Point at *b* . . . . . .
CURVE *C*—Biquadratic with Double Point at *c* x x x x x x x x x x
CURVE *F*—Biquadratic with Double Point at *f* — — — — — — —
CURVE F_1—Biquadratic with Double Point at f_1 —— —— —— ——
CURVE *H*—Kiepert Hyperbola ——— . ——— . ———

In addition, these points fall into two triads abc and $ff_1\ \infty$ which are apolar. If abc and $ff_1\infty$ are to be apolar, we must have

$$3abc-(f+f_1+\infty)(ab+ac+bc)$$
$$+(a+b+c)(ff_1+f\infty+f_1\infty)-3ff_1\infty=0.$$

Then the sum of all quantities multiplying ∞ must equal zero, or

$$-(ab+ac+bc)+(a+b+c)f'+f_1)-3ff_1 \text{ should equal zero.}$$

Replacing $a+b+c$ by s_1 and $ab+ac+bc$ by s_2 we write:

$$(9) \qquad -s_2+s_1(f+f_1)-3ff_1 \text{ should equal zero.}$$

f and f_1 may be expressed in terms of a, b, and c as follows:

$$f=s_1/3-\overline{v}_1v_2/3\overline{v}_2, \qquad f_1=s_1/3-v_1\overline{v}_2/3\overline{v}_1,$$

where $v_1=a+\omega b+\omega^2c$ and $v_2=a+\omega^2b+\omega c$ (ω being one of the complex cube roots of unity.)
We have now

$$f+f_1=2s_1/3-(\overline{v}_1^2v_2+\overline{v}_2^2v_1)/3\overline{v}_1v_2,$$
$$ff_1=s_1^2/9+v_1v_2/9-s_1/9(\overline{v}_1^2v_2+\overline{v}_2^2v_1/\overline{v}_1\overline{v}_2).$$

Substituting these values in equation (9) we have

$$-s_2+2s_1^2/3-(s_1/3)[(\overline{v}_1^2v_2+\overline{v}_2^2v_1)/\overline{v}_1\overline{v}_2]$$
$$-s_1^2/3-v_1v_2/3+(s_1/3)[(\overline{v}_1^2v_2+\overline{v}_2^2v_1)/\overline{v}_1\overline{v}_2]$$

should equal zero.

$$s_1^2/3-s_2-v_1v_2/3 \text{ should equal zero.}$$

Now

$$v_1v_2=(a+\omega^2b+\omega c)(a+\omega b+\omega^2c)=s_1^2-3s_2.$$

Then we have

$$s_1^2/3-s_2-s_1^2/3+s_2=0;$$

which shows that the triads abc and $ff_1\infty$ are apolar. This gives us

THEOREM 2. *The three vertices of a triangle are a set of points which is apolar to the set consisting of the two Fermat points of the triangle and the point infinity.*

II.

We now take, on a curve K (namely, a biquadratic which cuts itself at right angles), the double point and five other points, and derive the condition

under which the biquadratic having a double point at each and passing through the others, will be rectangular. If we choose K to be a biquadratic cutting itself at right angles at infinity, we know that this is a rectangular hyperbola, and since the other five points must lie on this curve, we see that we may take as their co-ordinates $x_i = m_i$ $y_i = 1/m_i$ $(i = 1, 2, 3, 4, 5)$. Here x and y are Cartesian co-ordinates, and are not complex.

If we wish to write the biquadratic passing through one of the points, say x_1y_1, which also passes through ∞, we need only write the equation of a circular cubic passing through x_1y_1. This will be of the form

$$(10) \quad \begin{aligned}&[(x-x_1)^2+(y-y_1)^2]\,[a(x-x_1)+b(y-y_1)]\\&+c(x-x_1)^2+d(x-x_1)(y-y_1)+e(y-y_1)^2\\&+f(x-x_1)+g(y-y_1)=0,\end{aligned}$$

(a, b, and c are real, arbitrary constants, and not the a, b, and c of the previous section.) When $x_1 = m_1$ and $y_1 = 1/m_1$ this cuts the hyperbola when

$$(11) \quad \begin{aligned}&[(m-m_1)^2+(1/m-1/m_1)^2]\,[a(m-m_1)+b(1/m-1/m_1)]\\&+c(m-m_1)^2+d(m-m_1)(1/m-1/m_1)\\&+e(1/m-1/m_1)^2+f(m-m_1)+g(1/m-1/m_1)=0.\end{aligned}$$

Since this is to have a double point at m_1, $1/m_1$ and cut itself at right angles there, we must have $f = g = 0$ and $e = -c$. Putting these values in equation (11), simplifying, and removing the factor $(m - m_1)^2$, we have

$$(12) \quad \begin{aligned}&[m^2m_1^2+1]\,[amm_1(m-m_1)-b(m-m_1)]\\&\qquad+cm^3m_1^3-dm^2m_1^2-cmm_1=0.\end{aligned}$$

This is obviously a quartic in m, the coefficient of m^4 being am_1^3. We may accordingly write equation (12) as

$$\begin{aligned}&[m^2m_1^2+1]\,[amm_1(m-m_1)-b(m-m_1)]+cm^3m_1^3\\&\quad-dm^2m_1^2-cmm_1\equiv am_1^3(m-a_1)(m-a_2)(m-a_3)(m-a_4),\end{aligned}$$

pass through the points m_2, $1/m_2$; m_3, $1/m_3$; m_4, $1/m_4$ and m_5, $1/m_5$, where a_1, a_2, a_3 and a_4 are the roots of the quartic. But we wish our curve to Accordingly, we may substitute m_2, m_3, m_4 and m_5 for a_1, a_2, a_3, and a_4 in the last equation, which then becomes

$$(13) \quad \begin{aligned}&[m^2m_1^2+1]\,[amm_1(m-m_1)-b(m-m_1)]+cm^3m_1^3\\&\quad-dm^2m_1^2-cmm_1\equiv am_1^3(m-m_2)(m-m_3)(m-m_4)(m-m_5).\end{aligned}$$

We are trying to find the conditions under which the biquadratic having a double point at m_1, $1/m_1$ and passing through the four other points m_2, $1/m_2$; m_3, $1/m_3$; m_4, $1/m_4$; m_5, $1/m_5$ and the point at infinity will be

rectangular. This will amount to finding some relation on m_1, m_2, m_3, m_4 and m_5. In (13) we have a relation involving all five m's, but it also involves the arbitrary constants a, b, c and d. We must get rid of these constants in some way. To do this, we take equation (13) and let $m = m_1$, $mm_1 = i$ and $mm_1 = -i$ successively, thus obtaining the three equations

$$(14) \qquad c(m_1^6 - m_1^2) - dm_1^4 = am_1^3(m_1^4 - m_1^3\sigma_1 + m_1^2\sigma_2 - m_1\sigma_3 + \sigma_4),$$

$$(15) \qquad -2ci + d = (a/m_1)(i^4 - i^3m_1\sigma_1 + i^2m_1^2\sigma_2 - im_1^3\sigma_3 + m_1^4\sigma_4),$$

$$(16) \qquad 2ci + d = (a/m_1)(i^4 + i^3m_1\sigma_1 + i^2m_1^2\sigma_2 + im_1^3\sigma_3 + m_1^4\sigma_4),$$

where σ_1, σ_2, σ_3 and σ_4 are the four symmetric functions of m_2, m_3, m_4 and m_5.

Solving the simultaneous equations (15) and (16) we obtain $c = a/2(m_1^2\sigma_3 - \sigma_1)$ and $d = a/m_1(1 - m_1^2\sigma_2 + m_1^4\sigma_4)$. Substituting these back in equation (14), we find that each term contains a as a factor, so that this may be cancelled out, leaving an expression which involves only m_1 and the symmetric functions of the other four m's. This expression is, after some reductions have been made,

$$(17) \qquad \sigma_1 + m_1^2\sigma_3 - 2m_1\sigma_4 - 2m_1 = 0.$$

This equation involves m_1 in a way different from that in which the remaining four m's are involved. This is because we started out by letting our biquadratic have a double point at m_1, $1/m_1$. But we wish to treat all the m's alike; i. e., we wish to find the condition under which the biquadratic having a double point at *any* m and passing through the others will be rectangular. So we must find a relation which will treat all the m's alike. To do this, we introduce the symmetric functions of m_1, m_2, m_3, m_4 and m_5, which are denoted by s_1, s_2, s_3, s_4 and s_5, and which are related to the σ's as follows:

$$s_1 = m_1 + \sigma_1, \qquad s_3 = m_1\sigma_2 + \sigma_3, \qquad s_5 = m_1\sigma_4.$$
$$s_2 = m_1\sigma_1 + \sigma_2, \qquad s_4 = m_1\sigma_3 + \sigma_4,$$

Also, we have

$$\sigma_1 = s_1 - m_1, \qquad \sigma_3 = (s_4m_1 - s_5)/m_1^2, \qquad \sigma_4 = s_5/m_1.$$

Putting these into equation (17) and simplifying, we have:

$$(18) \qquad m_1(s_4 - 3) + s_1 - 3s_5 = 0.$$

Equating coefficients of like powers of m_1 gives

$$(19) \qquad s_4 = 3, \qquad s_1 = 3s_5.$$

Equations (19) give the condition which we sought; it proves to be two

relations on m_1, m_2, m_3, m_4 and m_5; if we desire to have just one condition, we may effect this by substituting for the s's their values in terms of the σ's; this gives: $m_1\,\sigma_3 + \sigma_4 = 3$ and $m_1 + \sigma_1 = 3m_1\,\sigma_4$. Elimination of m_1 from these two equations gives

$$\sigma_1\sigma_3 - 10\sigma_4 + 3\sigma_4^2 + 3 = 0. \tag{20}$$

This is a relation on m_2, m_3, m_4 and m_5, but it will hold for any four of the m's. For instance, we might have started working with a biquadratic having a double point at m_2, and then we should have arrived at a relation (20) involving m_1, m_3, m_4 and m_5. So (20) holds for any four of the m's.

Equation (20) is fundamental in this work, so we state

THEOREM 3. *If we take, on a biquadratic which cuts itself at right angles at infinity, the point infinity and five other points whose co-ordinates are* m_1, $1/m_1$; m_2, $1/m_2$; m_3, $1/m_3$; m_4, $1/m_4$ *and* m_5, $1/m_5$, *then the condition that the biquadratic having a double point at* m_1, $1/m_1$ *and passing through the other five points shall be rectangular is:* $\sigma_1\sigma_3 - 10\sigma_4 + 3\sigma_4^2 + 3 = 0$, *where the* σ's *are the symmetric functions of* m_2, m_3, m_4 *and* m_5.

III.

In order to get a geometrical interpretation of the condition given in equation (20), we investigate the connection between our six points and Neuberg's cubic curve for three points.

This curve is defined as follows: starting out with three points $x_1\,y_1$, $x_2\,y_2$, $x_3\,y_3$, Neuberg's cubic is the locus of points $x_4\,y_4$ satisfying the condition.*

$$\Delta' \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23} + \lambda_{14} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31} + \lambda_{24} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12} + \lambda_{34} & 1 \end{vmatrix} = 0,$$

where λ_{23} is the squared distance between the points x_2y_2 and x_3y_3, and similarly for the otherλ's. This curve passes through the points x_1y_1, x_2y_2 and x_3y_3. In other words, if four points lie on a Neuberg cubic, Δ' vanishes for them.

If we subject the four points involved in Δ' to an inversion with center at x_5y_5 and radius equal to k, we get

$$\Delta' = k^{12} I_2 / \lambda_{15}^2 \lambda_{25}^2 \lambda_{35}^2 \lambda_{45}^2,$$

* See the note by Professor Morley, referred to above.

where

$$I_2 \equiv \begin{vmatrix} \lambda_{23}\lambda_{14} & \lambda_{23}\lambda_{15}\lambda_{45} + \lambda_{14}\lambda_{25}\lambda_{35} & 1 \\ \lambda_{31}\lambda_{24} & \lambda_{31}\lambda_{25}\lambda_{45} + \lambda_{24}\lambda_{35}\lambda_{15} & 1 \\ \lambda_{12}\lambda_{34} & \lambda_{12}\lambda_{35}\lambda_{45} + \lambda_{34}\lambda_{15}\lambda_{25} & 1 \end{vmatrix} = 0.$$

If $\Delta' = 0$ is the original Neuberg cubic, then $I_2 = 0$ is a new curve, which is the locus of a point $x_4 y_4$, the other four points being given. The curve whose equation is $I_2 = 0$ is a biquadratic; it is a covariant biquadratic curve of the four points x_1, y_1; x_2, y_2; x_3, y_3 and x_5, y_5. This curve, which may be denoted by C_2, is the "inverted Neuberg cubic with respect to one of the points (say x_5, y_5) of any three of the remaining four (say x_1, y_1; x_2, y_2 and x_3, y_3). The fifth point x_4, y_4 is the variable point which traces out the locus." * If five points lie on an inverted Neuberg cubic, I_2 must vanish for them.

It should be noticed that while the Neuberg cubic seems to involve only four points, it really involves five, since it is a biquadratic on infinity. So it involves the four points which are explicitly contained in it, plus the point infinity. The inverted Neuberg cubic, or better, covariant biquadratic C_2 involves five points explicitly, since, on inversion, infinity goes into $x_5 y_5$, and this appears in the equation.

After this digression on the Neuberg cubic and the covariant biquadratic C_2, we discuss the connection between these curves and our set of six points.

If we wish four of our points, say m_2, $1/m_2$; m_3, $1/m_3$; m_4, $1/m_4$, and m_5, $1/m_5$ to lie on a Neuberg cubic, we must have

$$\begin{vmatrix} \left[(m_3 - m_4)^2 + \dfrac{(m_4 - m_3)^2}{m_4^2 m_3^2}\right]\left[(m_2 - m_5)^2 + \dfrac{(m_5 - m_2)^2}{m_5^2 m_2^2}\right] & & \\ (m_3 - m_4)^2 + \dfrac{(m_4 - m_3)^2}{m_4^2 m_3^2} + (m_2 - m_5)^2 + \dfrac{(m_5 - m_2)^2}{m_5^2 m_2^2} & & 1 \\ \left[(m_4 - m_2)^2 + \dfrac{(m_2 - m_4)^2}{m_2^2 m_4^2}\right]\left[(m_3 - m_5)^2 + \dfrac{(m_5 - m_3)^2}{m_3^2 m_5^2}\right] & & \\ (m_4 - m_2)^2 + \dfrac{(m_2 - m_4)^2}{m_2^2 m_4^2} + (m_3 - m_5)^2 + \dfrac{(m_5 - m_3)^2}{m_3^2 m_5^2} & & 1 \\ \left[(m_2 - m_3)^2 + \dfrac{(m_3 - m_2)^2}{m_2^2 m_3^2}\right]\left[(m_4 - m_5)^2 + \dfrac{(m_5 - m_4)^2}{m_4^2 m_5^2}\right] & & \\ (m_2 - m_3)^2 + \dfrac{(m_3 - m_2)^2}{m_2^2 m_3^2} + (m_4 - m_5)^2 + \dfrac{(m_5 - m_4)^2}{m_4^2 m_5^2} & & 1 \end{vmatrix}$$

The problem of evaluating this determinant is quite interesting in itself, but since it involves many detailed operations which would be tedious for the

* Morley, *loc. cit.*, page 408.

reader, and which are unimportant for the purpose of this paper, we may omit them, and let it suffice to say that the determinant may finally be reduced to

$$(m_2m_3m_4m_5-1)^2\begin{vmatrix}(A-B)^2 & C & 1\\ (A-C)^2 & B & 1\\ (B-C)^2 & A & 1\end{vmatrix}+\begin{vmatrix}C^2(A-B)^2 & C & 1\\ B^2(A-C)^2 & B & 1\\ A^2(B-C)^2 & A & 1\end{vmatrix}=0,$$

where

$$A=m_2m_3+m_4m_5,\qquad B=m_2m_4+m_3m_5,\qquad C=m_2m_5+m_3m_4.$$

This gives

$$(\sigma_4-1)^2[-3(A-B)(B-C)(C-A)]$$
$$-(AB+AC+BC)(A-B)(B-C)(C-A)=0.$$

Then either

$$3(\sigma_4-1)^2+(AB+AC+BC)=0 \text{ or } (A-B)(B-C)(C-A)=0.$$

Making the second expression equal zero makes one of its factors zero. If, say, $A-B=0$, then $A=B$ and $m_2m_3+m_4m_5=m_2m_4+m_3m_5$. This makes either $m_5=m_2$ or $m_4=m_3$. We assumed our six points to be distinct, so this relation cannot hold. Therefore, we have

$$3(\sigma_4-1)^2+(AB+AC+BC)=0,\qquad 3(\sigma_4-1)^2+\sigma_1\sigma_3-4\sigma_4=0,$$
$$3\sigma_4^2-6\sigma_4-4\sigma_4+\sigma_1\sigma_3+3=0,\qquad \sigma_1\sigma_3-10\sigma_4+3\sigma_4^2+3=0.$$

Thus the condition that four of the points lie on a Neuberg cubic is exactly the same as the condition that the biquadratic having a double point at one of the six points and passing through the others shall be rectangular, since in each case the condition is

$$\sigma_1\sigma_3-10\sigma_4+3\sigma_4^2+3=0.$$

Since this holds for *any four* of our five points, we know that *any four* of them lie on a Neuberg cubic, or Δ' vanishes for any four of our five finite points. However, this really involves not four but *five* points. For when we are given a set of n points in the inversive plane, we have, in reality, $n+1$ points, for the point infinity may always be regarded as an additional point of the set. So this condition involves, in reality, five points of our set, namely, four of the five finite points and infinity. If we do not like to bother with the point infinity, we can invert with respect to a point a_6. So that now, instead of dealing with a set of points m_1, $1/m_1$; m_2, $1/m_2$; m_3, $1/m_3$; m_4, $1/m_4$; m_5, $1/m_5$ and infinity, we are dealing with a set of points a_1, a_2, a_3, a_4, a_5 and a_6, where $a_i (i=1\cdots 5)$ is the inverse of the point m_i, $1/m_i$

and a_6 is the inverse of infinity. The old condition for rectangularity will invert into a new condition, and this will be the condition that the biquadratic having a double point at one of the six (finite) points $a_1, \cdots, a_6$ and passing through the others shall be rectangular. The old condition involved four points, and said that any four of the five finite points in the old set lay on a Neuberg cubic; the new condition involves five points and says that any *five* of the points $a_1, \cdots, a_6$ lie on a covariant curve C_2 (since a Neuberg cubic inverts into a C_2), or better, that I_2 vanishes for any five of the six points.

Now, suppose we investigate more closely our set of points $a_1, \cdots, a_6$. Four of these, say a_2, a_3, a_4 and a_5 will determine a covariant curve C_2 which passes through a_2, a_3, a_4 and a_5. *Any point on this curve,* together with the points a_2, a_3, a_4 and a_5, will satisfy the condition $I_2 = 0$. We have already seen that any five of our set of points $a_1, \cdots, a_6$ satisfy the condition $I_2 = 0$. That is, I_2 will be zero for the points a_1, a_2, a_3, a_4 and a_5 and also for the set of points a_6, a_2, a_3, a_4 and a_5. From this it is evident that a_1 and a_6 lie on the curve determined by a_2, a_3, a_4 and a_5; that is, the C_2 determined by a_2, a_3, a_4 and a_5 passes through a_1 and a_6. In the same way we see that the C_2 determined by *any four* of the points is on the other two.

We may also put it as follows:

Choose a set of five of the points, say a_2, a_3, a_4, a_5 and a_6. Then

a_2, a_3, a_4, a_5 determine a C_2 which passes through a_1 and a_6,
a_2, a_3, a_4, a_6 determine a C_2 which passes through a_1 and a_5,
a_2, a_3, a_5, a_6 determine a C_2 which passes through a_1 and a_4,
a_2, a_4, a_5, a_6 determine a C_2 which passes through a_1 and a_3,
a_3, a_4, a_5, a_6 determine a C_2 which passes through a_1 and a_2.

It is to be noticed that all these C_2's pass through a_1. Thus we may say: If we choose a set of five of the six points, then, by choosing sets of four points from this set of five points, we determine five C_2's. All five of these C_2's then pass through the sixth point.

The results of the last three paragraphs are the most important ones arrived at in this paper. Accordingly, we re-state them, for better emphasis, as

THEOREM 4. *If we have a set of six points, which are such that the biquadratic having a double point at each, and passing through the others is rectangular, then the invariant I_2 vanishes for any five of them. The covariant curve C_2 determined by any four is on the other two. Moreover, if we choose a set of five of the six points, and set up the five covariant curves C_2 which are determined by sets of four points chosen from this set of five points, then all these five C_2's pass through the sixth point.*

IV.

It is proper to regard the six points and associated biquadratics, which we have studied so far, as the fundamental points and curves of a Geiser involution in the inversive plane.

In the inversive plane, circles play the rôle which lines play in the projective plane. A Cremona transformation in this plane is one which transforms circles into curves which behave like circles; i. e., into curves C_n* which are rational, which form a triply infinite system, and which are such that any two of them have as many *free* points of intersection as do two circles.

Professor Morley, starting from this definition,† has determined that there are four symmetric Cremona transformations in the inversive plane, and has given some of their properties. The transformations are:

A. One which transforms circles into circles,
B. One which transforms circles into *C_3's with four double points,*
C. One which transforms circles into *C_7's with six four-fold points,*
D. One which transforms circles into C_{15}'s *with seven eight-fold points.*

These transformations are all involutory.

Type C involves six points, and is therefore the one to which we shall confine our attention. This transformation sends a circle into a C_7 with six four-fold points. These six points are the fundamental points of the transformation. What are the fundamental curves? They are, of course, the transforms of the fundamental points. The transformation sends the C_7 on which a given fundamental point F is situated into a circle. Since the point F is of multiplicity four, a point P, in describing C_7, passes through F four times. The transform of P, called P', describes a circle, and this circle must meet the transform of F, which is a curve C_n, just four times. A circle and a C_n meet $2n$ times; therefore, $2n = 4$, $n = 2$, and the fundamental curve is a biquadratic, or C_2. As a matter of fact, it can be shown that the biquadratic corresponding to a given fundamental point has a double point at that point, and passes through the remaining five fundamental points. Thus our fundamental system consists of six points and six biquadratics, each biquadratic having a double point at one of the six points, and passing through the others.

To a general circle corresponds then a C_7 with four-fold points a_i

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* See explanatory note at end of paper.

† Unpublished report in mathematics seminary, Johns Hopkins University, December, 1926. The question as to whether a transformation so defined is one-to-one is not of interest here; it was, however, discussed by Professor Morley in the report mentioned here.

($i = 1, \cdots, 6$). When the circle is on a_1, the corresponding fundamental biquadratic A_1 is a factor of C_7; the circle then corresponds to the other factor, a C_5 with a double point at a_1 and triple points at the other a's.

When the circle is also on a_2, then A_2 factors out from C_5, and what corresponds to the circle is a C_3, with a_1, a_2 as simple points and $a_3\, a_4\, a_5\, a_6$ double points.

Finally, to the circle on a_1, a_2, a_3 corresponds the circle on a_4, a_5, a_6.

To what Cremona transformation in the projective plane does this transformation correspond? In the projective plane, a Cremona transformation transforms lines into curves which behave like lines. So we must see what the effect of our transformation on a line will be. In the inversive plane, a line is simply a circle on infinity. Let us put one of the fundamental points of our transformation at infinity. Then, by (1) above, the transform of a line will be a C_5 with infinity as a double point, and five triple points. But a C_5 with infinity as a double point is, projectively, an octavic with I and J as triple points. Thus we see that our transformation corresponds to one in the projective plane which sends a line into an octavic with seven triple points. This is the well-known Geiser involution, and by analogy we call Type C the Geiser involution in the inversive plane. We can now restate our main theorem thus:

There is a Geiser involution for which the six biquadratics are all rectangular. This requires three conditions on the six points.

Note on Curves in the Inversive Plane.

For the sake of those readers who are not familiar with the handling of curves in the inversive plane, I append the following note:

In this plane we use as co-ordinates $z = x + iy$ and $\bar{z} = x - iy$ (x and y being the ordinary Cartesian co-ordinates of a point.) These are variously called minimal, circular or absolute co-ordinates. A curve is given by an expression in z and $\bar{z}$, equated to zero. It is generally written in the form of a self-conjugate matrix such as

	1	z	z^2	z^3	z^4	$\cdots$	z^n	
1	P_0	a_1	a_2	a_3	a_4	$\cdots$	a_n	
$\bar{z}$	$\bar{a}_1$	P_1	b_1	b_2	b_3	$\cdots$	b_{n-1}	
$\bar{z}^2$	$\bar{a}_2$	$\bar{b}_1$	P_2	c_1	c_2	$\cdots$	c_{n-2}	
$\bar{z}^3$	$\bar{a}_3$	b_2	$\bar{c}_1$	P_3	d_1	$\cdots$	d_{n-3}	
$\bar{z}^4$	$\bar{a}_4$	$\bar{b}_3$	$\bar{c}_2$	$\bar{d}_1$	P_4			$= 0,$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$			
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$			$\cdot$	
$\bar{z}^n$	$\bar{a}_n$	$\bar{b}_{n-1}$	$\bar{c}_{n-2}$	$\bar{d}_{n-3}$			P_n	

where the p's are real and the other numbers are complex quantities and their conjugates. This is read by multiplying every element by the quantities heading the row and column in which the element stands and adding all such products together. Thus, the curve written above would start off as

$$P_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots + a_n z^n + \bar{a}_1 \bar{z} + P_1 z\bar{z} + b_1 z^2 \bar{z} + b_2 z^3 \bar{z} + \cdots = 0.$$

A curve is always named after the highest powers of z and $\bar{z}$ occurring in it. For instance, if the term of highest power is $p_n z^n \bar{z}^n$ the curve is called a bi-n-ic, and is denoted by C_n. It is easily seen that this curve is, projectively, a $2n$-*ic*, with an n-fold point at I and an n-fold point at J, since if we express $p_n z^n \bar{z}^n$ in Cartesian co-ordinates it becomes $p_n (x^2 + y^2)^n$ and this is the leading term of the curve when written in the ordinary way. The simplest curve is the C_1

$$\begin{array}{c|cc} & 1 & z \\ \hline 1 & P_0 & a_1 \\ \bar{z} & \bar{a}_1 & P_1 \end{array} = 0,$$

or $p_0 + a_1 z + \bar{a}_1 \bar{z} + p_1 z\bar{z} = 0$, which is a circle, since it is a conic on I and J. The next simplest is the C_2 or biquadratic

$$\begin{array}{c|ccc} & 1 & z & z^2 \\ \hline 1 & P_0 & a_1 & a_2 \\ \bar{z} & \bar{a}_1 & P_1 & b_1 \\ \bar{z}^2 & \bar{a}_2 & \bar{b}_1 & P_2 \end{array} = 0,$$

or

$$P_0 + a_1 z + \bar{a}_1 \bar{z} + a_2 z^2 + \bar{a}_2 \bar{z}^2 + P_1 z\bar{z} + b_1 z^2 \bar{z} + \bar{b}_1 \bar{z}^2 z + P_2 z^2 \bar{z}^2 = 0$$

which is projectively a quartic with I and J as double points.

If, in a C_n, the coefficient of $z^n \bar{z}^n$ is zero, the C_n passes through infinity. If, in addition, the coefficients of $z^n\ \bar{z}^{n-1}$ and $\bar{z}^n\ \bar{z}^{n-1}$ are both zero, the curve has a double point at infinity. Thus, writing the curve in matrix form, we see that by striking off the element in the lower right-hand corner, we make the curve pass through infinity. This causes the curve to be, projectively speaking, a $(2n-1)$ic, with I and J as $(n-1)$ fold points, for our highest term will be of the form

$$\begin{aligned} k z^n \bar{z}^{n-1} + \bar{k} \bar{z}^n z^{n-1} &= z^{n-1} \bar{z}^{n-1} [z(k_1 + ik_2) + \bar{z}(k_1 - ik_2)] \\ &= z^{n-1} \bar{z}^{n-1} [(k_1 + ik_2)(x + iy) + (k_1 - ik_2)(x - iy)] \\ &= (x^2 + y^2)^{n-1} [2k_1 x - 2k_2 y]. \end{aligned}$$

Taking off in addition the next two elements in the lower right-hand corner makes infinity a double point, and makes the curve equivalent to a curve in the projective plane of order $2n - 2$, with $(n - 2)$-fold points at I and J. We see immediately that a circle on infinity is a straight line, while a C_2 on infinity is a circular cubic. A C_2 with a double point at infinity is a conic.

In general, we may say that starting in the lower right-hand corner of the matrix and striking off p diagonals in succession makes our C_n have a p-fold point at infinity; it is a curve in the projective plane of order $(2n - p)$ with $(n - p)$ fold points at I and J. Given a curve in the inversive plane whose highest terms are of the form $kz^p\bar{z}^q$ and $\bar{k}\bar{z}^pz^q$, where $p \neq q$, then the curve is a C_n, where n is the greater of the two integers p and q, which passes through infinity $p - q$ or $q - p$ times. It is, in the projective plane a curve of order $p + q$ which passes through I and J each either q or p times. Thus, a curve whose highest terms are of the form $z^2\bar{z}$ and $\bar{z}^2z$ is a C_2 with infinity as a simple point. It is projectively a curve of the third degree which passes simply through I and J; i. e., a circular cubic.

A C_n with an n-fold point at infinity is a curve in the projective plane of order n, not passing through I and J. Accordingly we may say that an ordinary curve of order n in the projective plane is, in the inversive plane, a C_n with an n-fold point at infinity. A curve of order n in the projective plane which passes through I and J each m times becomes, in the inversive plane, a C_{n-m} with infinity as $(n - 2m)$fold point, whose highest terms are of the form $z^{n-m}\bar{z}^m$ and $z^m\bar{z}^{n-m}$. Thus, a curve of order nine in the projective plane, which passes through I and J each three times, becomes a C_6 with infinity as a triple point, whose highest terms are of the form $z^6\bar{z}^3$ and $\bar{z}^6z^3$.

One more point concerning these curves must be called to the attention of the reader: A C_n and a C_m have in common $2mn$ points.* For the C_n, being a $2n$-ic with n-fold points at I and J, and the C_m being a $2m$-ic with m-fold points at I and J, intersect in $4mn$ points; but the intersections at I and J use up $2mn$ of these, so that, as a consequence, we may say that they have in common only $4mn - 2mn = 2mn$ points.

* The common points are either actual intersections or common image-pairs.

Concerning the Rational Curves $R_3^5(II)$ and R_2^4.*

By H. E. Arnold.

1. The rational space quintic curve, R_3^5, is given parametrically by four binary quintics

$$R_3^5: \qquad x_i = (\alpha_i t)^5, \qquad (i = 0, 1, 2, 3). \tag{1}$$

The pencil of apolar quintics

$$\rho(bt)^5 + \sigma(ct)^5 \tag{2}$$

has been called the fundamental involution, and the necessary and sufficient condition that a binary quintic represent a plane section of the curve is that it be apolar to the fundamental involution.

In general there is a unique quartic apolar to the fundamental involution, and therefore a rational space quintic possesses a unique quadrisecant line. But if the four conditions imposed on the binary quartic by making it apolar to the quintics (2) are not independent, then there is a pencil of quartics apolar to the fundamental involution. This means that the curve possesses an infinity of quadrisecant lines, and it is said to be of the second species.† We shall designate this type of curve by the symbol $R_3^5(II)$, and shall call the pencil of apolar quartics,

$$\lambda(\beta t)^4 + \mu(\gamma t)^4, \tag{3}$$

the "auxiliary involution." The curve lies on a quadric and the ∞' quadrisecant lines form one set of rulings.

The rational plane quartic curve is represented by three binary quartics

$$R_2^4: \qquad x_i = (\alpha_i t)^4, \qquad (i = 0, 1, 2), \tag{4}$$

and the fundamental pencil is the involution of quartics apolar to (4). The form (3) will be considered also as the fundamental pencil of R_2^4.

The two curves R_2^4 and $R_3^5(II)$ can therefore be regarded as being determined, to within projections, by a pencil of binary quartics, or by means of the unique binary sextic, Ψ, apolar to such a pencil.

Setting

$$p_{ij} = \begin{vmatrix} \beta_i & \beta_j \\ \gamma_i & \gamma_j \end{vmatrix}, \tag{5}$$

* Presented to the American Mathematical Society, October 27, 1928.

† Bertini, "Sulle Curve Gobbe Razionali Del. 5. Ordine," *Collectanea Mathematica in mem. D. Chelini.*

the combinants * of (3) are expressed in terms of these determinants. The invariants for various special types of the curve R_2^4 have been found † and in the following table we give the relationship of some of these forms to the quintic $R_3^5(II)$.

Conditions on Pencil of Quartics	Special Quadrisecant lines of $R_3^5(II)$	Quartic R_2^4 possesses
$I_4-(I_2'-36I_2)^2=0$	1 bitangent	Degenerate flex conic
$I_4-(I_2'-36I_2)^2=0$ $I_6-16I_2^3=0$	2 bitangents	Biflecnode
$I_4-(I_2'-36I_2)^2=0$ $I_6-16I_2^3=0$ $(qt)^2\equiv 0$	3 bitangents	3 biflecnodes
$I_6=0$	1 linear inflexion	1 cusp
$I_6=0$ $[(I_2'-4I_2)^2-I_4]^2$ $+128[I_2I_2'I_4-I_2(I_2'-4I_2)^3+28I_2^2I_4]=0$	2 linear inflexions	2 cusps
$I_6=0$ $[(I_2'-4I_2)^2-I_4]^2$ $+128[I_2I_2'I_4-I_2(I_2'-4I_2)^3+28I_2^2I_4]=0$ $I_4-64I_2(I_2'-4I_2)-(I_2'-4I_2)^2=0$	3 linear inflexions	3 cusps
Apolar sextic Ψ has triple root	1 four-point contact tangent	R_2^4 reduces to rational cubic ‡
Apolar sextic Ψ is cube of a quadratic	2 four-point contact tangents	R_2^4 reduces to conic

* The relations between the Morley and the Salmon combinants (Salmon, ***Modern Higher Algebra***, 4th Edition, pp. 219-221) are: $I_2=-B$; $I_2'=12B-A$; $I_4=R$; $I_6=D$; $(qt)^8=P$; $(st)^6=J$.

† Rowe, "Covariants and Invariants of the Rational Plane Quartic," ***Transactions of the American Mathematical Society***, Vol. 12, p. 304; Winger, "Self-Projective Rational Curves of the Fourth and Fifth Orders," ***American Journal of Mathematics***, Vol. 36, p. 59.

‡ Thomsen, "The Osculants of Plane Rational Quartic Curves," ***American Journal of Mathematics***, Vol. 32, p. 207.

2. The curve $R_3^5(II)$ for which I_2 vanishes is particularly interesting. The vanishing of I_2 is the condition that the pencil (3) be the first polars of a binary quintic. For the plane rational quartic $I_2 = 0$ is the triple point condition, and it is known that the canonizant of the quintic form gives the parameters of the triple point.

Let us take as the base quintic form

$$(6) \qquad (at)^5 = a_0t^5 + 5a_1t^4\tau + 10a_2t^3\tau^2 + 10a_3t^2\tau^3 + 5a_4t\tau^4 + a_5\tau^5.$$

The fundamental involution is readily obtained from the auxiliary involution, and we note that the five points of the curve which are given by the roots of (6) lie on a plane, which we shall call the plane π. Let us designate by $(at_1)(at)^4$ and $(at_2)(at)^4$, respectively, the first polars of (t_1, τ_1) and (t_2, τ_2) with respect to (6). It can easily be verified that

$$(7) \quad (t\tau_2 - t_2\tau)(at_1)(at)^4 - (t\tau_1 - t_1\tau)(at_2)(at)^4 = (t_1\tau_2 - t_2\tau_1)(at)^5.$$

But

$$(8) \qquad (t\tau_2 - t_2\tau)(at_1)(at)^4$$

and

$$(9) \qquad (t\tau_1 - t_1\tau)(at_2)(at)^4$$

are plane sections of the curve. Moreover, when the quintics giving three plane sections are linearly related the planes are in a pencil, and hence the planes (8) and (9) intersect on π. Thus if P_1 and P_2 are any two points of the curve, and q_1 and q_2 their polar quadrisecants, respectively, we have the planes P_1q_2 and P_2q_1 intersecting on the plane π.

If we let $R_3^5(II)$ be given by

$$(10) \qquad R_3^5(II) \begin{cases} x_0 = t[\lambda_1(\beta t)^4 + \mu_1(\gamma t)^4] \\ x_1 = \tau[\lambda_1(\beta t)^4 + \mu_1(\gamma t)^4] \\ x_2 = t[\lambda_2(\beta t)^4 + \mu_2(\gamma t)^4] \\ x_3 = \tau[\lambda_2(\beta t)^4 + \mu_2(\gamma t)^4], \end{cases}$$

where $\lambda_1(\beta t)^4 + \mu_1(\gamma t)^4$ and $\lambda_2(\beta t)^4 + \mu_2(\gamma t)^4$ are any two members of the auxiliary involution, then the curve lies on the quadric

$$(11) \qquad x_0x_3 - x_1x_2 = 0,$$

and the equation of the plane π is

$$(12) \qquad \mu_2x_0 - \lambda_2x_1 - \mu_1x_2 + \lambda_1x_3 = 0.$$

The equations of the plane sections (8) and (9) are given by

$$(13) \qquad (t_1\mu_2 - \tau_1\lambda_2)(\tau_2 x_0 - t_2 x_1) + (\tau_1\lambda_1 - t_1\mu_1)(\tau_2 x_2 - t_2 x_3) = 0$$

and

$$(14) \qquad (t_2\mu_2 - \tau_2\lambda_2)(\tau_1 x_0 - t_1 x_1) + (\tau_2\lambda_1 - t_2\mu_1)(\tau_1 x_2 - t_1 x_3) = 0,$$

respectively.

If C be the conic in which π cuts the quadric (11), and if S_1 and S_2 be the points in which C is met by the quadrisecants q_1 and q_2, respectively, then as P_2 moves along the curve while P_1 remains fixed, the planes (13) and (14) give rise to a pencil of lines through S_1. As S_2 is brought into coincidence with S_1 the line S_1S_2 becomes tangent to C, and we have the theorem: *If I_2 vanishes the plane through a point P_1 of the curve and its polar quadrisecant q_1 cuts the plane π in a line which is tangent to the conic C.* A simple geometric method for finding the pole of a given quadrisecant is thus afforded.

But the tangent line to C at S_1 lies in the plane of the two generators through that point, and hence *the conic C may be further characterized as being the locus of points through which pass a unisecant and a quadrisecant generator such that the point on the unisecant is the pole, with respect to $(at)^5$, of the quartic giving the quadrisecant.* The curve $R_3^5(II)$ itself may then be considered as being specialized in this way when the invariant I_2 vanishes.

If we let $t = 0$ be one of the roots of (6), then the polar of $(0, 1)$ gives the quadrisecant through this point. But $t = 0$ is then also a root of the polar quadrisecant, and hence one of the four points has moved up to the conic C. This gives the theorem: *At the five points whose parameters are the roots of the base quintic* (6), *the tangent line to the curve, the tangent to the conic, and the quadrisecant through the point considered are coplanar.*

3. In the theory of the rational plane quartic the undulation invariant I_4 was obtained as the eliminant of the two forms $(\beta t)^4$ and $(\gamma t)^4$ of the fundamental pencil. If we impose the condition that the I_4 of the auxiliary involution of $R_3^5(II)$ vanish, then each member of the system of quadrisecants would have a root in common. Since the quadrisecants form one system of generators of the quadric on which the curve lies, that surface would have to be a quadric cone. But then a plane through two of the elements of the cone would cut the quintic in seven points. Similarly, if $(\beta t)^4$ and $(\gamma t)^4$ had two roots in common a plane through two elements would meet $R_3^5(II)$ in six points. If, however, the two quartics have three roots in common the curve has a triple point at the vertex of the cone.* We

* Mueller, Dissertation, *Die Rationale Kurve Fünfter Ordnung im Fünf-, Vier-, Drei- und Zweidimensionalen Raum.* Leipzig (1910), p. 78.

conclude, therefore, that I_4 *cannot vanish for any* $R_3^5(II)$ *which does not have a triple point.*

The $R_3^5(II)$ with a triple point corresponds, then, to the R_2^4 with three undulations. As in the case of such a plane quartic the sextic Ψ for $R_3^5(II)$ is not a unique form, but any member of a net of sextics will serve to recover the curve.*

4. The $R_3^5(II)$ which possesses two four-point contact tangents is interesting because it sets up a null system. Taking the equations of such a curve as given by Colpitts,†

$$(15) \qquad R_3^5(II): \quad x_0 = t^5, \quad x_1 = t\tau^4, \quad x_2 = t^4\tau, \quad x_3 = \tau^5,$$

the equations in planes are found to be

$$(16) \qquad \xi_0 = 3\tau^5, \quad \xi_1 = 5t^4\tau, \quad \xi_2 = -5t\tau^4, \quad \xi_3 = -3t^5.$$

Hence it can be shown without difficulty that the five planes which osculate the curve at the points cut out by the plane $(\xi x) = 0$ meet in a point which itself lies on (ξx); dually, a point in space gives rise to a plane through itself.

5. A pencil of binary quartics has also been studied from the point of view of a pencil of line sections of R_2^4. In what follows we will make use of this interpretation.

Consider an R_2^4 with distinct nodes B, C, D, and the ∞^5 cubics C_2^3 which pass through B, C, D, and any fourth point A not on R_2^4. Each such cubic C_2^3 cuts R_2^4 in six more points, and we have therefore ∞^5 binary sextics. Such a system of cubics is determined by six linearly independent members, and likewise the involution of sextics is defined by the six corresponding sextics. But there is a unique sextic $f^6 = (dt)^6$ apolar to the six independent sextics. Any sextic apolar to f^6 is a member of the involution, and one cubic C_2^3 passes through the six points corresponding to the given sextic. The roots of f^6 give the contacts of the six cubics C_2^3 which have six-point contact with R_2^4.

Now any line through A, together with a conic on B, C, and D is a degenerate cubic C_2^3. This means that the four points on a line through A, with any two arbitrary points of the curve R_2^4, give a sextic apolar to f^6. In other words, the quartics obtained from the pencil of lines through A are apolar to f^6. Hence the theorem: *The roots of the unique sextic apolar*

* Neelley, "Compound Singularities of the Rational Plane Quartic Curve," *American Journal of Mathematics*, Vol. 49, p. 391.

† Colpitts, "On Twisted Quintic Curves," *American Journal of Mathematics*, Vol. 29, p. 336.

to the pencil of quartics obtained from the line sections through a fixed point not on R_2^4 are the parameters of the contacts of the six cubics C_2^3 which pass through the nodes of R_2^4 and the fixed point and have six-fold contact with the curve.

Furthermore, when we examine the degenerate cubics C_2^3 which pass through one and two fixed points, respectively, of R_2^4, we obtain the theorem:

The four points on a line through A are such that any three of them form the canonizant of the first polar of the fourth with respect of f^6; the second polars of f^6 with respect to two points on a line through A give harmonic sets.

We note particularly the case where the point E on the quartic, taken with the points A, B, C, D, determines a conic, which in turn cuts out another point of the curve. Calling this sixth point F, we see that the net of line sections of R_2^4 gives rise to quartics apolar to the second polars $(dt_E)(dt_F)(dt)^4$, where E and F are in an involution $I_{1,1}^2$ formed by the conics on A, B, C, D. Thus the quartics of the net of line sections are not apolar to a net of quartics,—the second polars of f^6—as might seem to be the case, but to the pencil of quartics obtained from the quadratic involution. We may take as the base quartics of the pencil the pure second polars of f^6 with respect to the two points E_1 and E_2 at which conics on A, B, C, D have contact with R_2^4. Designating this pencil by

$$\lambda(dt_{E_1})^2(dt)^4 + \mu(dt_{E_2})^2(dt)^4, \tag{17}$$

and noting that the net of line sections of R_2^4 is apolar to (17), we see that (17) is the fundamental pencil of the curve. The net of second polars of f^6 therefore contains the fundamental pencil. We can thus say that

The ∞^2 points of the plane give rise to a net of binary sextics, f^6, each including among its second polars the fundamental pencil of the quartic R_2^4.

On Varieties of Three Dimensions with Six Right Lines through Each Point.

By CHARLES H. SISAM.

Among surfaces, it is a well known and quite useful theorem that if, through each point of the surface, there pass just two distinct right lines lying on the surface, then the surface is a quadric. It is the purpose of this paper to prove an analogous theorem for varieties of three dimensions; namely, *if through each point of the variety there pass just six right lines which lie on the variety, then the variety is a cubic hypersurface in four dimensions.*

Denote by V_3 such a variety of three dimensions. Since each rectilinear generator at a point P of V_3 is a fourpoint tangent to V_3 at P, the variety has, at each of its points, six fourpoint tangents and lies in a space of four dimensions.* No three of the generators through a generic point can be coplanar; otherwise the quadric cone of three point tangents to V_3 would be composite and the six generators through that point would not be distinct.

Let a given generic point O on V_3 be taken as origin and the tangent S_3 to V_3 at this point be taken as the coördinate space $w = 0$. Let the equation of V_3 be given in each of the forms

$$(1) \qquad w = \phi_2(x, y, z) + \phi_3(x, y, z) + \phi_4(x, y, z) + \cdots$$

and

$$(2) \qquad w = A(x, y) + zB(x, y) + z^2C(x, y) + z^3D(x, y) + \cdots,$$

where ϕ_i is homogeneous of degree i in x, y, z and A, B, C, etc. are power series in x and y only.

We shall suppose the coördinate system chosen so that no generator in the neighborhood of the origin lies in a space $z = \text{const.}$ Let the six systems of rectilinear generators on V_3 be defined on the variety by the equations

$$(3) \qquad x = u + z\sigma_i(u, v) \quad y = v + zt_i(u, v) \qquad (i = 1, 2, 3, 4, 5, 6)$$

where

$$(4) \qquad \begin{aligned} \sigma_i(u_1v) &= \sigma_0^{(i)} + \sigma_1^{(i)} + \sigma_2^{(i)} + \sigma_3^{(i)} + \cdots \\ \tau_1(u_1v) &= t_0^{(i)} + t_1^{(i)} + t_2^{(i)} + t_3^{(i)} + \cdots \qquad (i = 1, 2, 3, 4, 5, 6) \end{aligned}$$

* Compare, e. g., the author, "On Three-spread Satisfying Four or More Homogeneous, Linear, Partial Differential Equations of the Second Order," *American Journal of Mathematics*, Vol. 33 (1911), pp. 97-128.

in which $\sigma_0^{(i)}$, $t_0^{(i)}$ are constants and $\sigma_j^{(i)}$, $t_j^{(i)}$ are homogeneous of degree j in u and v. From (2) and (3) we have, further, for the generators of each system

$$w = A(u,v) + z[B(u,v) + A_u\sigma_i + A_v t_i] \tag{5}$$

and the restrictive conditions on the coefficients in (2)

$$C + B_u\sigma_i + B_v t_i + (\tfrac{1}{2}!)[A_{uu}\sigma_i^2 + 2A_{uv}\sigma_i t_i + A_{vv}t_i^2] \equiv 0 \tag{6a}$$
$$D + C_u\sigma_i + C_v t_i + (\tfrac{1}{2}!)[B_{uu}\sigma_i^2 + \cdots] + (\tfrac{1}{3}!)[A_{uuu}\sigma_i^3 + \cdots] \equiv 0 \tag{6b}$$
$$E + D_u\sigma_i + D_v t_i + (\tfrac{1}{2}!)[C_{uu}\sigma_i^2 + \cdots] + \cdots \equiv 0 \tag{6c}$$
$$F + E_u\sigma_i + E_v t_i + (\tfrac{1}{2}!)[D_{uu}\sigma_i^2 + \cdots] + \cdots \equiv 0 \tag{6d}$$

etc.

Each rectilinear generator at the origin lies in $w = 0$ and also satisfies the equations

$$\phi_2 = 0 \qquad \phi_3 = 0 \qquad \phi_4 = 0 \qquad \text{etc.}$$

In particular, since the six distinct intersections of $\phi_2 = 0$ $\phi_3 = 0$ lie on $\phi_4 = 0$, we have

$$\phi_4 = \alpha_1\phi_3 + \beta_2\phi_2, \tag{7}$$

where α_1 is linear and β_2 is quadratic in x, y, z.

Consider, now, the cubic hypersurface

$$w = \phi_2 + \phi_3 + \alpha_1(w - \phi_2) + \beta_2 w + \gamma_0 w(w - \phi_2) + \beta_1 w^2 + \beta_0 w^3 = 0 \tag{8}$$

where α_1 and β_2 are the forms defined in (7), the quantities β_0, γ_0 are constants, as yet undefined, and β_1 is an undetermined linear form in x, y, z.

We shall show that the quintic cone

$$\phi_5 - \alpha_1\phi_4 - \beta_2\phi_3 = 0 \tag{9}$$

in $w = 0$ has each of the generators $w = 0$ $\phi_2 = 0$ $\phi_3 = 0$ of V_3 at the origin as a double generator. It will then follow that this quintic has $\phi_2 = 0$ as a component and that the residual cubic component passes through each of these generators and is thus of the form $\gamma_0\phi_3 + \beta_1\phi_2 = 0$, so that

$$\phi_5 \equiv \alpha_1\phi_4 + \beta_2\phi_3 + \gamma_0\phi_2\phi_3 + \beta_1\phi_2^2. \tag{10}$$

If, in (8), we take for γ_0 and β_1 the quantities defined by (10), the expansion in series of the cubic hypersurface (8) will coincide with (1) to the terms of ϕ_5 inclusive.

To show that the cone (9) has the given lines as generators, we take the coördinate system so that any one of the generators at the origin is $w = x = y = 0$ and that

$$\phi_2 \equiv xz - y^2,$$
$$\phi_3 \equiv yz^2 + zB_2(x, y) + A_3(x, y),$$

where B_2 and A_3 denote respectively the second degree terms of B and the third degree terms of A in (2). Let, also

$$\alpha_1(xyz) \equiv \alpha_{00}z + \alpha_{10}x + \alpha_{01}y,$$
$$\beta_2(x_1y_1z) \equiv \beta_{00}z^2 + \beta_{10}xz + \beta_{01}yz + \cdots;$$

then, from (7),

$$\phi_4 \equiv \beta_{00}xz^3 + \alpha_{00}yz^3 + \cdots.$$

Since $x = y = w = 0$ is a generator, ϕ_5 vanishes for $x = y = 0$. Let

$$\phi_5 \equiv a_5xz^4 + b_5yz^4 + \cdots.$$

From (6a) and (6b), since $\sigma_{00} = t_{00} = 0$, we have

(11) $$\sigma_1 \equiv -v, \qquad t_1 \equiv -(\beta_{00}u + \alpha_{00}v),$$

where σ_1 and t_1 are the first degree terms of σ and t respectively for the system that contains the line $x = y = w = 0$.

From (6c) we now have

$$\phi_5(u, v, 1) \equiv \alpha_{00}\beta_{00}u + (\beta_{00} + \alpha_{00}^2)v + \cdots.$$

But, in coördinates u, v, 1, we have

$$\alpha_1\phi_4 + \beta_2\phi_3 \equiv \alpha_{00}\beta_{00}u + (\beta_{00} + \alpha_{00}^2)v + \cdots.$$

Hence the cone $\phi_5 - (\alpha_1\phi_4 + \beta_2\phi_3) = 0$ in $w = 0$ has the generator $x = y = w = 0$ as a double line, and, since this is any one of the six generators at the origin, ϕ_5 is of the form (10).

Next, we shall show that the cone

(12) $$\phi_6 - [\alpha_1\phi_5 + \beta_2\phi_4 + \gamma_0\phi_2\phi_4 + \gamma_0\phi_3^2 + 2\beta_1\phi_2\phi_3] = 0$$

in $w = 0$ has each of the generators $w = \phi_2 = \phi_3 = 0$ of V_3 at the origin as a triple line from which it will follow at once that

(13) $$\phi_6 \equiv \alpha_1\phi_5 + \beta_2\phi_4 + \gamma_0\phi_2\phi_4 + \gamma_0\phi_3^2 + 2\beta_1\phi_2\phi_3 + \beta_0\phi_2^3.$$

If, in (9), we take for β_0 the value determined by (13), then the expansion of the cubic hypersurface coincides with that of V_3 as given by (1) to the terms of ϕ_6 inclusive.

If we denote by A_2, B_2, C_2, etc., the terms of A, B, C, etc., of second degree in u and v, we find, from (6a) and (6b)

$$\sigma_2 \equiv -[C_2 + B_{2u}\sigma_1 + B_{2v}t_1 + A_2(\sigma_1t_1)]$$
$$t_2 \equiv -[D_2 + C_{2u}\sigma_1 + C_{2v}t_1 + B_2(\sigma_1t_1)]$$

from which, using the values of σ_1, t_1 determined by (11), we have σ_2, t_2

determined uniquely in terms of the coefficients of ϕ_2, ϕ_3, ϕ_4, ϕ_5. From (6c), (6d), and (6e), we now have

$$E_2 \equiv -[D_{1u}\sigma_2 + D_{1v}t_2 + D_{2u}\sigma_1 + D_{2v}t_1 + C_2(\sigma_1 t_1)]$$
$$F_1 \equiv [E_{1u}\sigma_1 + E_{1v}t_1], \qquad G_0 = 0.$$

These equations fix the coefficients of the terms of $\phi_6(u, v, 1)$ of degree not more than two in u and v uniquely in terms of those of ϕ_2, ϕ_3, ϕ_4, ϕ_5. It follows that these coefficients are identical with those of the corresponding expansion of the cubic (8). But, for (8), the sextic corresponding to (12) is identical with $-\beta_0\phi_2^3 = 0$ and thus has a triple generator along $x = y = w = 0$. It follows that the given sextic (12) has each generator of $w = \phi_2 = \phi_3 = 0$ as a triple generator and thus is identical with a constant times $\phi_2^3 = 0$. Hence ϕ_6 is of the form given by equation (13).

Finally, the terms of $\phi_7(u, v, 1)$ involving u and v to not more than the third power are determined uniquely by those of $\phi_2, \cdots, \phi_6$. In fact, from (6a), (6b), we obtain

$$\sigma_3 \equiv -[C_3 + B_{2u}\sigma_2 + B_{3u}\sigma_1 + B_{2v}t_2 + B_{3v}t_1 - 2t_1t_2 + \tfrac{1}{2}(A_{3uu}\sigma_1^2 + 2A_{3uv}\sigma_1 t_1 + A_{3vv}t_1^2)]$$
$$t_3 \equiv -[D_3 + C_{2u}\sigma_2 + C_{3u}\sigma_1 + C_{2v}t_2 + C_{3v}t_1 + B_{2uu}\sigma_1\sigma_2 + B_{2uv}(\sigma_1 t_2 + \sigma_2 t_1) + B_{2vv}t_1t_2 + \tfrac{1}{2}(B_{3uu}\sigma_1^2 + 2B_{3uv}\sigma_1 t_1 + B_{3vv}t_1^2) + A_3(\sigma_1, t_1)].$$

From these equations, σ_3 and t_3 are uniquely determined in terms of the coefficients of $\phi_2, \cdots, \phi_6$.

From (6c), (6d), (6e) and (6f), we find

$$E_3 \equiv -[D_{1u}\sigma_3 + D_{2u}\sigma_2 + D_{3u}\sigma_1 + D_{1v}t_3 + D_{2v}t_2 + D_{3v}t_1 + C_{2uu}\sigma_1\sigma_2 + C_{2uv}(\sigma_1 t_2 + \sigma_2 t_1) + C_{2vv}t_1t_2 + \tfrac{1}{2}(C_{3uu}\sigma_1^2 + 2C_{3uv}\sigma_1 t_1 + C_{3vv}t_1^2) + A_3(\sigma_1 t_1)]$$
$$F_2 \equiv -[E_{1u}\sigma_2 + E_{2u}\sigma_1 + E_{1v}t_2 + E_{2v}t_1 + D_2(\sigma_1 t_1)]$$
$$G_1 \equiv -[F_{1u}\sigma_1 + F_{1v}t_1] \qquad H_0 = 0.$$

These equations fix the terms of $\phi_7(u, v, 1)$ of degree three or less in u and v terms of those of $\phi_2, \cdots, \phi_6$. It follows that these are identical with the corresponding terms of ϕ_7', which arises from the seventh degree terms of the cubic (8) so that $\phi_7(x, y, z) - \phi_7'(x, y, z) = 0$ has $x = y = w = 0$ as a fourfold line. Since this holds for each generator $w = \phi_2 = \phi_3 = 0$, we have $\phi_7 \equiv \phi_7'$. Thus the cubic (8) has contact with (1) to the terms in ϕ_7 inclusive. Since this contact holds at an origin chosen generically on V_3, it follows that the given variety coincides with the cubic hypersurface.

The Oscillation of a Sequence.*

By Wallie Abraham Hurwitz.

1. *Introduction.* For any given sequence (x) of real or complex numbers $x_1, x_2, x_3, \cdots$ let

$$\Omega(x) = \lim_{m\to\infty,\, n\to\infty} \sup |x_m - x_n|$$

be termed the *oscillation* of the sequence. In order that (x) be convergent it is necessary and sufficient that $\Omega(x) = 0$. The oscillation of a sequence may be considered as measuring the deviation of the sequence from convergence, just as the oscillation of a function at a point may be considered as measuring the deviation of the function from approach to a limit at the point. As defined, $\Omega(x)$ may be a (finite) number $\geqq 0$ or the symbol $+\infty$.

In order that $\Omega(x)$ be finite, it is necessary and sufficient that (x) be bounded.

If (x) is bounded, obviously $\Omega(x)$ is finite. On the other hand suppose that $\Omega(x)$ is finite, and let $Q > \Omega(x)$. There exists an integer q such that when $m \geqq q$, $n \geqq q$, then $|x_m - x_n| < Q$. In particular, for $n \geqq q$, $|x_n - x_q| < Q$ and $|x_n| < Q + |x_q|$; therefore the greatest of the q numbers

$$|x_1|, |x_2|, \cdots, |x_{q-1}|, Q + |x_q|$$

is an upper bound of all $|x_n|$.

For a bounded sequence, $\Omega(x)$ may also be defined as the maximum of all distances between pairs of limiting values of (x). For a bounded real sequence

$$\Omega(x) = \lim_{n\to\infty} \sup x_n - \lim_{n\to\infty} \inf x_n. \tag{1}$$

2. *Effect of regular transformation.* By the familiar Silverman-Toeplitz theorem, in order that the transformation

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k \tag{A}$$

be regular, i. e., transform every convergent sequence (x) into a sequence (y) having the same limit, it is necessary and sufficient that

* Read before the American Mathematical Society, December 27, 1929.

$$(2)\qquad \begin{cases} \text{(I)} & \textit{for each } k,\ \lim_{n\to\infty} a_{n,k} = 0; \\ \text{(II)} & \lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1; \\ \text{(III)} & \textit{for all } n,\ \sum_{k=1}^{n} |\, a_{n,k} \,| \textit{ is bounded.} \end{cases}$$

The object of the present paper is to consider the change effected on the oscillation of a sequence by a regular linear transformation of the form (A), and specifically to insure that

$$(3)\qquad \Omega(y) \leqq \Omega(x).$$

The problem has an obvious relation to a result obtained in an earlier paper *; it was shown that if the elements of (A) are real, and only real sequences (x) are considered, then in order that a regular transformation (A) may be such that

$$(4)\qquad \lim_{n\to\infty} \sup y_n \leqq \lim_{n\to\infty} \sup x_n$$

and

$$(5)\qquad \lim_{n\to\infty} \inf y_n \geqq \lim_{n\to\infty} \inf x_n$$

whenever (x) is bounded, it is necessary and sufficient that

$$(6)\qquad \text{(IV)}\quad \lim_{n\to\infty} \sum_{k=1}^{n} |\, a_{n,k} \,| = 1.$$

In fact, for a bounded real sequence, it is clear that (IV), being sufficient for (4) and (5), is sufficient for (3). But (3) does not imply both of (4), (5), so that the theorem mentioned does not show that (IV) is necessary for (3). A slight modification of the proof given would establish this necessity, even for the complex case; but in the complex case, on the other hand, (4) and (5) are meaningless and therefore cannot be used in order to show sufficiency. The whole question is therefore now discussed independently; the following theorem will be proved:

In order that a regular transformation (A) *may be such that* $\Omega(y) \leqq \Omega(x)$ *for every sequence* (x), (IV) *is necessary and sufficient.*

3. *Proof of Necessity.* To prove the necessity of (IV) we suppose that (I), (II), (III) hold, but (IV) fails, and exhibit a sequence (x) for which $\Omega(y) > \Omega(x)$, thereby establishing a contradiction.† If

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* W. A. Hurwitz, *Proceedings of the London Mathematical Society*, Series 2, Vol. 26 (1927), pp. 231-248.

† The construction of the sequence (x) involves only minor modifications of the proof previously mentioned; *loc. cit.*, pp. 238-239.

$$\lim_{n\to\infty} \sup \sum_{k=1}^{n} | a_{n,k} | > 1,$$

then for some $\alpha > 1$, there will exist values of n arbitrarily great for which $\sum_{k=1}^{n} | a_{n,k} | > \alpha$. Choose n_1 so that

$$\sum_{k=1}^{n_1} | a_{n_1,k} | > \alpha.$$

By (1), $\sum_{k=1}^{n_1} | a_{n,k} | \to 0$ as $n \to \infty$; we can therefore choose n_2 so that

$$\sum_{k=1}^{n_1} | a_{n_2,k} | < 1/2, \qquad \sum_{k=1}^{n_2} | a_{n_2,k} | > \alpha.$$

In general, choose n_p so that

$$\sum_{k=1}^{n_{p-1}} | a_{n_p,k} | < 1/p, \qquad \sum_{k=1}^{n_p} | a_{n_p,k} | > \alpha;$$

then also

$$\sum_{k=n_{p-1}+1}^{n_p} | a_{n_p,k} | > \alpha - 1/p.$$

Now define *

$$x_k = \begin{cases} \operatorname{sgn} a_{n_1,k}, & k \leqq n_1; \\ (-1)^{p-1} \operatorname{sgn} a_{n_p,k}, & n_{p-1} < k \leqq n_p. \end{cases}$$

Then for all k, $| x_k | \leqq 1$, and $\Omega(x) \leqq 2$. We find that

$$\begin{aligned} y_{n_{p+1}} - y_{n_p} &= \sum_{k=1}^{n_{p+1}} a_{n_{p+1},k} x_k - \sum_{k=1}^{n_p} a_{n_p,k} x_k \\ &= \sum_{k=1}^{n_p} a_{n_{p+1},k} x_k - \sum_{k=1}^{n_{p-1}} a_{n_p,k} x_k + \sum_{k=n_p+1}^{n_{p+1}} a_{n_{p+1},k} x_k - \sum_{k=n_{p-1}+1}^{n_p} a_{n_p,k} x_k \\ &= \sum_{k=1}^{n_p} a_{n_{p+1},k} x_k - \sum_{k=1}^{n_{p-1}} a_{n_p,k} x_k \\ &\qquad + (-1)^p \sum_{k=n_p+1}^{n_{p+1}} | a_{n_{p+1},k} | + (-1)^p \sum_{k=n_{p-1}+1}^{n_p} | a_{n_p,k} | . \end{aligned}$$

In the last expression the absolute values of the first two terms are together not greater than

$$\sum_{k=1}^{n_p} | a_{n_{p+1},k} | + \sum_{k=1}^{n_{p-1}} | a_{n_p,k} | < 1/(p+1) + 1/p < 2/p;$$

the absolute values of the last two terms (which are real and of like sign) are together greater than

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* Here $\operatorname{sgn} z = | z | / z$ or 0 according as $z \neq 0$ or $z = 0$; hence in all cases $z \operatorname{sgn} z = | z |$.

$$\{\alpha - [1/(p+1)]\} + [\alpha - (1/p)] > 2\alpha - 2/p.$$

Hence

$$| y_{n_{p+1}} - y_{n_p} | > 2\alpha - 4/p.$$

Thus

$$\Omega(y) = \lim_{m\to\infty,\, n\to\infty} \sup | y_m - y_n | \geqq \lim_{p\to\infty} \sup | y_{n_{p+1}} - y_{n_p} |$$
$$\geqq \lim_{p\to\infty} (2\alpha - 4/p) = 2\alpha > 2 \geqq \Omega(x),$$

and $\Omega(y) > \Omega(x)$.

4. *Proof of Sufficiency.* It will now be shown that if (I), (II), (III), (IV) hold, then $\Omega(y) \leqq \Omega(x)$. In case $\Omega(x) = +\infty$, no proof is necessary. Let $\Omega(x) = \omega$ be finite; since (x) must be bounded, let $| x_n | \leqq X$ for all n; let $\sum_{k=1}^{n} | a_{n,k} | = A_n$. For $\epsilon > 0$ there is an integer p such that when $\mu > p$, $\nu > p$, then $| x_\mu - x_\nu | < \omega + \epsilon$. If $m > p$, $n > p$,

$$y_m - y_n = \sum_{k=1}^{m} a_{m,k}x_k - \sum_{k=1}^{n} a_{n,k}x_k$$
$$= \sum_{k=1}^{p} (a_{m,k} - a_{n,k})x_k + \sum_{k=p+1}^{m} a_{m,k}x_k - \sum_{k=p+1}^{n} a_{n,k}x_k.$$

The absolute value of the first term of the last expression is not greater than

$$2X\left(\sum_{k=1}^{p} | a_{m,k} | + \sum_{k=1}^{p} | a_{n,k} |\right),$$

which, by (I), has the limit 0 as m and n become infinite. Thus

$$\Omega(y) = \lim_{m\to\infty,\, n\to\infty} \sup \left| \sum_{k=p+1}^{m} a_{m,k}x_k - \sum_{k=p+1}^{n} a_{n,k}x_k \right|.$$

Furthermore

$$\sum_{k=p+1}^{m} a_{m,k}x_k - \sum_{k=p+1}^{n} a_{n,k}x_k = \sum_{\mu=p+1}^{m} a_{m,\mu}x_\mu - \sum_{\nu=p+1}^{n} a_{n,\nu}x_\nu$$
$$= \left(\sum_{\mu=p+1}^{m} a_{m,\mu}x_\mu\right)\left(1 - \sum_{\nu=p+1}^{n} a_{n,\nu}\right) - \left(\sum_{\nu=p+1}^{n} a_{n,\nu}x_\nu\right)\left(1 - \sum_{\mu=p+1}^{m} a_{m,\mu}\right)$$
$$+ \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} a_{m,\mu}a_{n,\nu}(x_\mu - x_\nu).$$

But

$$\left| \left(\sum_{\mu=p+1}^{m} a_{m,\mu}x_\mu\right)\left(1 - \sum_{\nu=p+1}^{n} a_{n,\nu}\right) \right| \leqq XA_m \left| 1 - \sum_{\nu=p+1}^{n} a_{n,\nu} \right|,$$

$$\left| \left(\sum_{\nu=p+1}^{n} a_{n,\nu}x_\nu\right)\left(1 - \sum_{\mu=p+1}^{m} a_{m,\mu}\right) \right| \leqq XA_n \left| 1 - \sum_{\mu=p+1}^{n} a_{m,\mu} \right|,$$

each of which, by (I), (II), and (IV), has the limit 0 as m and n become infinite. Thus

$$\Omega(y) = \lim_{m\to\infty,\ n\to\infty} \sup \Big| \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} a_{m,\mu} a_{n,\nu} (x_\mu - x_\nu) \Big| .$$

Finally,

$$\Big| \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} a_{m,\mu} a_{n,\nu} (x_\mu - x_\nu) \Big| \leqq \sum_{\mu=p+1}^{m} \sum_{\nu=p+1}^{n} |a_{m,\mu}|\,|a_{n,\nu}|\,(\omega+\epsilon)$$

$$\leqq (\omega+\epsilon) A_m A_n,$$

so that, by (IV)

$$\Omega(y) \leqq \omega + \epsilon.$$

Since this holds for each $\epsilon > 0$, it follows that $\Omega(y) \leqq \omega = \Omega(x)$.

5. *Variations of the Theorem.* For the special sequence used in § 3 in order to prove the necessity of (IV), $\Omega(x)$ is finite; hence

In order that the regular transformation (A) *may be such that* $\Omega(y) \leqq \Omega(x)$ *for every sequence* (x) *such that* $\Omega(x)$ *is finite,* (IV) *is necessary and sufficient.*

The Silverman-Toeplitz theorem holds if we restrict ourselves to real transformations of real sequences; also if every element of (A) is real, then the sequence (x) chosen in § 3 is real; hence

In order that the regular transformation (A) *with real elements may be such that* $\Omega(y) \leqq \Omega(x)$ *for every real sequence* (x), (IV) *is necessary and sufficient.*

6. *An Interpretation.* It may not be superfluous to point out that when (II) is satisfied, (IV) is equivalent to the requirement that in the sum of absolute values of elements of a row of (A), the total contribution of the pure imaginary parts and of those real parts which are negative, is negligible. In order to state this more precisely and prove it, we note the following lemma:

For any real numbers u_k, v_k,

$$\Big(\sum_{k=1}^{n} u_k\Big)^2 + \Big(\sum_{k=1}^{n} v_k\Big)^2 \leqq \Big[\sum_{k=1}^{n} (u_k^2 + v_k^2)^{1/2}\Big]^2.$$

For $n = 2$, this follows at once by rationalization; for $n > 2$, it can be shown by induction or repeated application of the case $n = 2$.

In this formula write $u_k = |b_{n,k}|$, $v_k = |c_{n,k}|$, where $a_{n,k} = b_{n,k} + ic_{n,k}$ and $b_{n,k}$, $c_{n,k}$ are real:

$$\Big(\sum_{k=1}^{n} |b_{n,k}|\Big)^2 + \Big(\sum_{k=1}^{n} |c_{n,k}|\Big)^2 \leqq \Big(\sum_{k=1}^{n} |a_{n,k}|\Big)^2. \tag{7}$$

Suppose (II) holds; then

$$\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k} = 1 \tag{8}$$

and

$$\sum_{k=1}^{n} b_{n,k} \leqq \sum_{k=1}^{n} | b_{n,k} | \leqq \sum_{k=1}^{n} | a_{n,k} | . \tag{9}$$

If (IV) also holds, then by (IV), (8), and (9),

$$\lim_{n\to\infty} \sum_{k=1}^{n} | b_{n,k} | = 1; \tag{10}$$

and by (IV), (7), and (10),

$$\lim_{n\to\infty} \sum_{k=1}^{n} | c_{n,k} | = 0. \tag{11}$$

But (IV) can also be deduced from (II), (10), and (11), since

$$\sum_{k=1}^{n} a_{n,k} \leqq \sum_{k=1}^{n} | a_{n,k} | \leqq \sum_{k=1}^{n} | b_{n,k} | + \sum_{k=1}^{n} | c_{n,k} | .$$

Thus:

If (II) *holds,* (IV) *is equivalent to* (10) *and* (11). Now let

$$b^{+}_{n,k} = (b_{n,k} + | b_{n,k} |)/2, \quad b^{-}_{n,k} = (b_{n,k} - | b_{n,k} |)/2, \tag{12}$$

so that

$$b_{n,k} = \begin{cases} b^{+}_{n,k} & \text{if} \quad b_{n,k} > 0, \\ b^{-}_{n,k} & \text{if} \quad b_{n,k} < 0; \end{cases}$$
$$b^{+}_{n,k} \geqq 0; \qquad b^{-}_{n,k} \leqq 0.$$

Then

$$b_{n,k} = b^{+}_{n,k} + b^{-}_{n,k}; \quad | b_{n,k} | = b^{+}_{n,k} - b^{-}_{n,k}. \tag{13}$$

From (10) and (8), by (12), we find

$$\lim_{n\to\infty} \sum_{k=1}^{n} b^{+}_{n,k} = 1, \quad \lim_{n\to\infty} \sum_{k=1}^{n} b^{-}_{n,k} = 0; \tag{14}$$

while from (14), by (13), we could deduce (10) and (8); from (8) and either equation of (14), by (13), we could deduce (10). Thus:

(II) *and* (IV) *are equivalent to* (11) *and* (14). *If* (II) *holds, then* (IV) *is equivalent to* (11) *and either equation of* (14).

Determination of All the Groups of Order 64.

By G. A. MILLER.

INTRODUCTION.

Among all the groups whose orders do not exceed 100 those of order 64 present by far the greatest difficulties in view of the very large number of distinct groups of this order. The groups of every other order within this limit have been published either separately or as special cases of well known general categories. In fact, since $64 = 2^6$ these groups are also included among those of order p^6, p being a prime number. This general category is the subject of a Paris thesis, 1904, by M. Potron, but not all the possible groups of order 64 appear therein, and the more general methods of treatment which are employed there naturally involve many difficulties which do not present themselves in this special category. It may be of interest to tabulate here the number of groups of every order not greater than 100 whenever there is more than one group of the same order.

Order	4	6	8	9	10	12	14	16	18	20	21	22	24	25	26	27
Number	2	2	5	2	2	5	2	14	5	5	2	2	15	2	2	5
Order	28	30	32	34	36	38	39	40	42	44	45	46	48	49	50	52
Number	4	4	51	2	14	2	2	14	6	4	2	2	52	2	5	5
Order	54	55	56	57	58	60	62	63	64	66	68	70	72	74	75	76
Number	15	2	13	2	2	13	2	4	294	4	5	4	50	2	3	4
Order	78	80	81	82	84	86	88	90	92	93	94	96	98	99	100	
Number	6	52	15	2	15	2	12	10	4	2	2	230	5	2	16	

The total number of groups whose orders do not exceed 100 is therefore 1074 and for each of 37 of these orders there is only one group.

1. *Abelian groups.* An abelian group of order p^m, p being a prime number, whose invariants are $p^{m_1}, p^{m_2}, \cdots, p^{m_\lambda}$ is said to be of type $(m_1, m_2, \cdots, m_\lambda)$, and there is only one such group if $m_1, m_2, \cdots, m_\lambda$ represent positive numbers such that $m_1 + m_2 + \cdots + m_\lambda = m$, and such a group is completely defined by its type. Hence there are exactly eleven groups of order 2^6, which correspond respectively to the following partitions of 6 with respect to addition: 6, $5+1$, $4+2$, $4+1+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$, $1+1+1+1+1+1$. It is known that every non-abelian group of order 64 contains an invariant abelian subgroup of order 16, but it does not necessarily contain an abelian subgroup of order 32.* We may therefore divide the non-abelian

* Miller, Blichfeldt, Dickson; *Finite Groups* (1916), p. 120.

groups of order 64 into two categories which are composed respectively of all those groups which involve at least one abelian subgroup of index 2 and of all those which do not have this property. The former category will be considered first.

2. *General theorems relating to non-abelian groups which contain an abelian subgroup of index* 2. If H represents an abelian subgroup of index 2 under the group G, then G can be obtained by adjoining to H an operator s which transforms H according to an automorphism of order 2 and has its square in H, and any operator which satisfies these two conditions gives rise to a group whose order is twice the order of H when it is adjoined to H. In order to determine these automorphisms of order 2 it is desirable to note that every commutator which arises from an automorphism of order 2 of any group whatever corresponds to its inverse under this automorphism, and that the commutators which arise from an automorphism of an abelian group constitute a subgroup of this abelian group.

Since every commutator which results from an automorphism of order 2 gives rise to a commutator which is the inverse of the square of the former commutator, it results that, when the order of a commutator arising from an automorphism of order 2 of an abelian group is odd, then every generator of this commutator which appears in this abelian group and has an order involving no prime factors besides those of the order of the commutator must give rise under the same automorphism to a commutator whose order is equal to the order of this generator. As this commutator must correspond to its inverse under this automorphism, there results the following theorem: *If a commutator of odd order results from an automorphism of order* 2 *of an abelian group, and if s is any generator of this commutator whose order involves no prime factor besides those which divide the order of this commutator, then an operator whose order is equal to that of s corresponds to its inverse under the same automorphism.* From this theorem it results directly that in an automorphism of order 2 of a cyclic group of odd prime power order every operator must correspond to its inverse, and hence there is only one possible automorphism of order 2 of such a cyclic group.

When a commutator arising from an automorphism of order 2 of an abelian group is of even order, this commutator will give rise to a commutator whose order is one-half of the order of the former commutator. In particular, when a commutator of order 2 arises from such an automorphism, it must correspond to itself under this automorphism. In general, a generator of a commutator of even order arising from such an automorphism must give rise to a commutator whose order is equal to one-half the order of this generator

whenever the order of this generator involves no prime factors except those which appear in the order of the commutator. From this it results that when the abelian group is cyclic and of order 2^m, then there is one automorphism of order 2 which gives rise to a commutator of order 2, while in the other possible automorphisms of order 2 at least one-half of the operators of this cyclic group must correspond to their inverses. There can therefore be only two such other automorphisms. Since every abelian group admits an automorphism in which every operator corresponds to its inverse, and such an automorphism of a cyclic group of order 2^m can not be the square of another automorphism of this group, it results from the given theorems that the group of isomorphisms of this cyclic group is of type $(\alpha, 1)$. This is a new proof of this well known fact and illustrates the usefulness of the theorems noted above.

In view of the fact that the multiplying group can often be used to simplify the study of the automorphisms of an abelian group, we note here a few more properties of this group. It is especially useful to know some of the properties of the multiplying group which corresponds to the transform of a given automorphism under the group of isomorphisms. It may first be noted that if under an automorphism of an abelian group the multiplier of a given operator s is t, then under the inverse of this automorphism t^{-1} is the multiplier of ts. In particular, *the multiplying group corresponding to an automorphism is the same as that which corresponds to the inverse of this automorphism.* A necessary and sufficient condition that the multiplier of an operator s under a given automorphism is the inverse of the multiplier of s under the inverse of this automorphism is that this multiplier is invariant under this automorphism. In general, if the multipliers of s corresponding to two automorphisms of an abelian group are t_1 and t_2, then the multiplier of s in the product of these two automorphisms, in order, is $t_3 t_2 t_1$, where t_3 is the multiplier of t_1 under the second automorphism.

The preceding theorems may be illustrated by a consideration of all the possible automorphisms of order 2 of the abelian group H of order 2^α and of type $(\alpha - 1, 1)$, $\alpha > 4$. The commutator subgroup corresponding to such an automorphism must be cyclic since a non-cyclic commutator subgroup would involve all the operators of order 2 contained in H and hence all these operators would correspond to themselves in this automorphism. The quotient group corresponding to a subgroup involving all the operators of order 2 contained in H is obviously cyclic. There are three non-conjugate operators of order 2 in the group of isomorphisms of H which correspond to commutator subgroups of order 2. When the commutator subgroup is of order 4, it is

not contained in a cyclic subgroup of order 8, and hence the two corresponding automorphisms of order 2 are conjugate.

When the order of the commutator subgroups corresponding to an automorphism of order 2 exceeds 4 it must be $2^{\alpha-2}$ since at least one-half of the operators of H must correspond to their inverses in such an automorphism in view of the fact that an operator which is the square of an operator of highest order must correspond to its inverse therein. There are two cyclic subgroups of order $2^{\alpha-2}$ in H and each of these is a commutator subgroup arising from an automorphism of order 2. When the commutator subgroup of this order is generated by the square of an operator of highest order contained in G and the corresponding central is non-cyclic, every operator of H corresponds either to its inverse or to its $2^{\alpha-2}-1$ power. These two automorphisms are not conjugate under the group of isomorphisms of H, but the two automorphisms of order 2 which arise when the central is cyclic and the commutator subgroup remains unchanged are conjugate. When the commutator subgroup is the other cyclic subgroup of order $2^{\alpha-2}$, the four possible automorphisms are conjugate in pairs. Hence the following theorem: *The group of isomorphisms of the abelian group of order 2^α and of type $(\alpha-1, 1)$, $\alpha > 4$, involves fifteen operators of order 2; three of these are invariant and the other twelve are conjugate in pairs. Hence there are nine such automorphisms which have the property that no two of them are conjugate under the group of isomorphisms.*

A very simple illustration of the general theorems noted above is furnished by the operations of order 2 in the group of isomorphisms of the abelian group of order 2^α and of type $(1, 1, 1\cdots)$. In this case all the commutators must correspond to themselves in such an automorphism, and hence the number of such non-conjugate automorphisms is equal to the largest integer which does not exceed $\alpha/2$. When the abelian group is of order 2^α and of type $(2, 1, 1\cdots)$, the commutator subgroup which corresponds to an automorphism of order 2 must again appear in the central under the automorphism, and the number of the independent generators of this commutator subgroup may be any natural number which does not exceed $\alpha/2$. When the commutator subgroup is of order 2 and $\alpha > 3$, there are always four such automorphisms which are not conjugate under the group of isomorphisms. The largest number of non-conjugate automorphisms of order 2 corresponding to a given order of the commutator subgroup is obviously 5, and this number is always attained when this order exceeds two and its square is less than one-half of the order of the group. When this square is equal to one-half of the order of the group, there are four such automorphisms, and when it is

equal to the order of the group, there are two such automorphisms. This gives rise to the following theorem: *The number of non-conjugate operators of order 2 in the group of isomorphisms of the abelian group of order* 2^m *and of type* $(2, 1, 1 \cdots)$ *is one if* $m = 2$, *three if* $m = 3$, *and if* $m > 3$ *and is even it is* $1 + 5(m-2)/2$, *while when* m *is odd it is* $3 + 5(m-3)/2$.

3. *Groups which involve either the cyclic group of order 32 or the abelian group of order 32 and of type* $(4, 1)$. It is well known that there are always four non-abelian groups of order 2^m, $m > 3$, which involve the cyclic group of order 2^{m-1}. As these four groups are so well known, we shall not consider their properties here. It was noted in the preceding section that when H is the abelian group of order 32 and of type $(4, 1)$ there are nine non-conjugate automorphisms of order 2, and hence we proceed to consider the possible groups of order 64 which may correspond to these automorphisms.

When the commutator subgroup is the characteristic subgroup of order 2 and the central is cyclic, there is only one group which does not involve an operator of order 32, while there are two groups when this central is non-cyclic. When the commutator subgroup is a non-characteristic subgroup of order 2, the central must be non-cyclic and there are two additional groups. Hence five distinct groups of order 64 result when the commutator subgroup is of order 2. Each of these five groups contains three abelian subgroups of index 2 in view of the obvious theorem that a necessary and sufficient condition that a non-abelian group of order p^m which involves an abelian subgroup of index p contains $p + 1$ abelian subgroups of this index is that the commutator subgroup is of order p.

When the commutator subgroup of H is of order 4, the central must be cyclic and there is only one corresponding group of order 64. When the commutator subgroup is of order 8, the central may be either cyclic or non-cyclic and the commutator subgroup may be either of the two cyclic subgroups of order 8. Hence there are four such automorphisms. There are three groups when each operator of H corresponds to its inverse, and there are two groups when these operators correspond to their seventh powers. When the central is non-cyclic and the commutator subgroup is the other cyclic subgroup of order 8, there are two additional groups. Hence there are seven groups when the central is non-cyclic and the commutator subgroup is of order 8. When the central is cyclic, it is the subgroup of order 4 which cannot be generated by an operator of order 8 contained in H. There are two groups when the commutator subgroup is the cyclic group which is generated by an operator of order 16, while there is only one group when the commutator subgroup is the other group of order 8. Hence there are 16

groups of order 64 which contain the abelian subgroup of type (4, 1) but no operator of order 32.

4. *Groups which involve the abelian group of type* (3, 2) *but no abelian group of order* 32 *which involves an operator of order* 16. In view of the theorem that if a non-abelian group of order p^m contains more than one abelian subgroup of index p, then each of its operators appears in at least one of its $p + 1$ abelian subgroups of this index, it is clear that the restriction imposed on the groups under consideration is equivalent to saying that when the commutator subgroup is of order 2, these groups cannot contain an operator of order 16. There are four non-conjugate automorphisms of order 2 in which the corresponding commutator subgroup is of order 2. When the central involves operators of order 8 and the commutator subgroup is the characteristic subgroup of order 2, there is only one group, but there are two groups when the commutator subgroup is not characteristic. When the central is of type (2, 2), two groups correspond to each of the two non-conjugate automorphisms but only three of these are distinct, and hence there are six groups when H is of type (3, 2) and the commutator subgroup is of order 2. When the commutator subgroup is the non-cyclic group of order 4, the central is of type (2, 1) and there are two non-conjugate automorphisms. Two groups correspond to one of these, while only one group results from the other. Hence three groups of order 64 result from the non-cyclic commutator subgroup of order 4.

When the commutator subgroup is one of the two cyclic groups of order 4 whose operators cannot be used as an independent generator of H, then there are two non-conjugate automorphisms of order 2. In one of these, exactly half of the operators of H correspond to their inverses, and this gives rise to three distinct groups of order 64, while the other gives rise to two additional groups of this order. Only one automorphism of order 2 corresponds to the other cyclic group of order 4 as a commutator subgroup, and this gives rise to two additional groups of order 64. Hence the cyclic commutator subgroups of order 4 give rise to seven groups of order 64 when the central is non-cyclic. It gives rise to two such groups when the central is cyclic. One of these two groups involves operators of order 16 which do not appear in an abelian subgroup of order 32.

It remains to consider the case when the commutator subgroup is of order 8 and hence it is non-cyclic. The central must therefore be the non-cyclic subgroup of order 4 contained in H, and the commutator subgroup must be composed of the operators in H which are squares. There are three groups in which all the operators of H correspond to their inverses, and two

additional ones when all of these operators correspond to their third powers. The remaining two automorphisms in which all the operators of order 4 correspond to their inverses are conjugate and there are two additional groups which correspond to one of these automorphisms. There are only two non-conjugate automorphisms of order 2 when not all the operators of order 4 correspond to their inverses. In one of these one-half of the operators of order 8 correspond to their inverses while the rest correspond to their third powers. Two groups of order 64 arise from this automorphism, while only one group results from the remaining automorphisms. Hence ten distinct groups result when the commutator subgroup is of order 8, and there are 28 groups which result from the automorphisms considered in this section and are not simply isomorphic with the groups resulting from the automorphisms considered in the preceding sections.

5. *Groups which involve the abelian group of type* $(3, 1, 1)$ *but no abelian subgroup of index* 2 *which has less than three independent generators.* Since a group of order 64 which contains the abelian group of type $(3, 1, 1)$ cannot contain any other abelian subgroup of order 32 when the order of its commutator subgroup exceeds 2, it is only necessary to exclude the groups from the enumeration of the present section in which one of the two other abelian subgroups contains just two independent generators. The group of type $(3, 1, 1)$ contains seven subgroups of index 2. One of these is of type $(2, 1, 1)$, and the other six are of type $(3, 1)$ and are conjugate under the group of isomorphisms of H. When the central is of type $(2, 1, 1)$, there are two groups whose commutator subgroup is the characteristic subgroup of order 2, and there are three additional groups whose commutator subgroup is a non-characteristic subgroup of order 2. When the central is of type $(3, 1)$, there is obviously only one additional group. Hence there are 6 groups of order 64 when the commutator subgroup is of order 2.

When the commutator subgroup is the non-cyclic group of order 4, it must appear in the central of H, and hence this central must be of type $(2, 1)$. There are therefore two non-conjugate such automorphisms of order 2. In one of these, four operators of order 4 correspond to their inverses and this gives rise to two groups, while the other gives rise to only one group. Hence three groups of order 64 result when the commutator subgroup is the non-cyclic group of order 4. When the commutator subgroup is the cyclic group of order 4 which is generated by an operator of order 8 and the central is of type $(1, 1, 1)$, every operator of H corresponds either to its inverse or to its third power. In the former case there are three groups while there are two such groups in the latter. When the commutator subgroup remains un-

changed but the central is of type $(2, 1)$, then half of the operators of H correspond to their inverses while the rest correspond to their third powers. There are three such groups. When the commutator subgroup is another cyclic subgroup of order 4 and the central is of type $(1, 1, 1)$, there are three groups and there are two additional groups when the central is of type $(2, 1)$. Hence 16 distinct groups result when the commutator subgroup is of order 4.

It remains to consider the possible groups when the commutator subgroup is of order 8. As this must be non-cyclic, the central must be the non-cyclic group of order 4 which involves the characteristic subgroup of order 2 and an independent generator of this order. The commutator subgroup must involve this central and the subgroup of order 4 which is generated by an operator of order 8. Hence at least 8 operators must correspond to their inverses and there are two non-conjugate automorphisms. In one of these, 16 operators correspond to their inverses, while in the other, eight operators correspond to their third powers. Hence three groups result when the commutator subgroup is of order 8 and the total number of groups resulting from the automorphisms of this section is 25.

6. *Groups of order* 64 *which contain an abelian subgroup of order* 32 *but no operator of order* 8 *which appears in such a subgroup.* If H is of type $(2, 2, 1)$ and the central is of type $(2, 2)$ there is only one group. The three abelian subgroups of index 2 contained in this group are of the same type. When the central is of type $(2, 1, 1)$ and the commutator subgroup is the characteristic subgroup of order 2 contained in the central, there are two groups. When this commutator subgroup is the square of an operator of order 4 in H which is not in the central, there are three groups, and when it is another subgroup of order 2 contained in H, there are two additional groups. Hence there are eight groups of order 64 which contain the abelian subgroup of type $(2, 2, 1)$ and two other abelian subgroups which do not involve an operator of order 8.

When the commutator subgroup is the non-cyclic group of order 4, the central must include the characteristic subgroup of this order contained in H. There are three such groups in which every operator of H corresponds to its inverse. If the commutator subgroup and the central remain the same, but only half of the operators of H correspond to their inverses, there are again three groups, while there are two additional groups when none of the operators corresponds to its inverse. When the central remains the same, but the non-cyclic commutator subgroup of order 4 is changed, there are three groups when half the operators of H correspond to their inverses, and there are two groups when this is not the case. When the central of G is of type $(2, 1)$ and

the commutator subgroup is the non-cyclic group of order 4, there are three groups in which half the operators of H correspond to their inverses, and two groups arise from the other possible automorphism.

A cyclic commutator subgroup of order 4 implies that the central is of type $(2, 1)$, and hence it results that there are two such groups. Therefore there are twenty groups of order 64 which involve the abelian group of type $(2, 2, 1)$ and give rise to a commutator subgroup of order 4 with respect to this abelian group. Suppose that in an abelian group of order 2^m an operator of order 2^a gives rise in an automorphism of order 2 to a commutator of order 2^a. The square of this operator is multiplied in this automorphism by an operator whose order is equal to this square. The central co-set to which this square belongs must involve an operator whose order is twice the order of this square since every commutator which arises from an automorphism of order 2 corresponds to its inverse in this automorphism. This proves the following theorem: *If an operator of order 2^a of an abelian group of order 2^m gives rise to a commutator of order 2^a in an automorphism of order 2, then the central under this automorphism must involve an operator of order 2^a which generates the same operator of order 2 as the commutator generates.* From this theorem it results directly that an automorphism of order 2 of the abelian group of type $(2, 2, 1)$ cannot give rise to a commutator subgroup of order 8 since this commutator subgroup would be of type $(2, 1)$.

If a group of order 64 involves the abelian subgroup of type $(2, 1, 1, 1)$ but no abelian subgroup of index 2 which has less than four independent generators, and its central is of type $(1, 1, 1, 1)$, there are three groups when every operator of H is transformed into its inverse and there is one group when the central is of type $(2, 1, 1)$. When the central is of type $(1, 1, 1, 1)$ but not every operator of an abelian subgroup of index 2 corresponds to its inverse, there are two more groups, so that there are six groups which have a commutator subgroup of order 2. If the order of the commutator subgroup exceeds 2, this subgroup cannot include any operator of order 4 and hence its order cannot exceed 4 in accord with the theorem noted in the preceding paragraph. If the central is of type $(1, 1, 1)$ and the commutator subgroup involves the characteristic subgroup of order 2, there are three groups when half of the operators of order 4 correspond to their inverses, and there are two groups in which none of these operators corresponds to its inverse.

When the central remains of type $(1, 1, 1)$, but the commutator subgroup does not involve the characteristic subgroup of order 2, there are two groups. When the central is of type $(2, 1)$ there are again two groups. Hence there are fifteen groups of order 64 which involve the abelian group of type

(2, 1, 1, 1) but no abelian subgroup of index 2 which has less than four independent generators. When all the abelian subgroups of index 2 in G are of type (1, 1, 1, 1, 1), the commutator subgroup must be of order 4 and there are two groups. In one of these, all the additional operators are of order 4, while only three-fourths of them have this property in the other. The total number of non-abelian groups of order 64 which involve an abelian subgroup of index 2 is therefore 118.

GROUPS CONTAINING NO ABELIAN SUBGROUP OF INDEX 2.

It was noted in § 1 that every group of order 64 contains an invariant abelian subgroup of order 16. The groups which remain to be considered may be divided into three distinct categories. The first of these is composed of all those which contain only one abelian subgroup of order 16. The second comprises all those which involve more than one such abelian subgroup but no two of them appear in a subgroup of order 32. The third category is composed of those which involve at least one subgroup of order 32 which contains three abelian subgroups of order 16. If such a group of order 64 contains only one abelian subgroup of order 16, this subgroup cannot be cyclic since such a subgroup must be transformed by the rest of the operators of the group according to a subgroup of order 4 in its group of isomorphisms, and every subgroup of order 4 in the group of isomorphisms of the cyclic group of order 2^m, $m > 2$, contains an operator of order 2 which transforms one-half of the operators of this cyclic group into themselves.

The group of isomorphisms of the abelian group of order 16 and of type (3, 1) can clearly be represented as a transitive group of degree 8 with respect to letters corresponding to its operators of order 8, and when it is thus represented it contains a subgroup of order 8 formed by a (2, 2) isomorphism between two regular non-cyclic groups of order 4. The remaining operators of this transitive group interchange the two cyclic subgroups of order 8 and are composed of four operators of order 2 and four of order 4. As the squares of the latter correspond to the automorphism in which every operator corresponds to its fifth power, it results that a group of order 64 which involves no abelian subgroup of order 16 besides the one of type (3, 1) cannot transform the operators of this subgroup according to a cyclic group of order 4. Two of the operators of order 2 in the group of isomorphisms of this abelian group which interchange its cyclic subgroups of order 8 are commutative with all of its operators which are not of highest order, and this is also the case of the product of the remaining two operators of order 2 which interchange these cyclic subgroups. It therefore results that if a group of order 64 con-

tains only one abelian subgroup of order 16, this subgroup cannot be of type (3, 1).

7. *Groups containing only one abelian subgroup of order* 16. In what precedes we used the symbol H to represent an abelian subgroup of index 2 but in this and the next section it will be convenient to use this symbol to represent the single abelian subgroup of order 16 contained in G since the properties of this subgroup will be frequently under consideration. From what precedes, it results directly that H must be one of the three groups whose types are (1, 1, 1, 1), (2, 1, 1), (2, 2). We shall first determine all the possible groups when H is of type (1, 1, 1, 1), and hence the group of isomorphisms of H is the alternating group of degree 8. This group involves two sets of conjugate operators of order 2. Under one of these half of the operators of H correspond to themselves and this operator corresponds to the substitution of degree 8 in the said alternating group. Hence it cannot appear in the group of order 4 according to which H is transformed under G. As this operator appears in one of the two sets of conjugate cyclic subgroups of order 4 contained in the group of isomorphisms of H it results that H appears in only one group of order 64 which transforms the operators of H according to a cyclic group of order 4.

There are two sets of conjugate non-cyclic groups of order 4 in the group of isomorphisms of H which correspond to subgroups of the alternating group which involve no substitution of degree 8. One of these sets corresponds to regular groups, and each group is composed of operators which are commutative with all the operators of the same subgroup of order 4 contained in H. Hence the operators of G cannot transform the operators of H according to this subgroup of order 4 since the resulting group would involve more than one abelian subgroup of order 16. The operators of G must therefore transform the operators of H according to a group of order 4 which corresponds to the non-cyclic group of order 4 and of degree 6 contained in the alternating group of degree 8. Hence there is also only one group of order 64 which transforms H according to a non-cyclic group of order 4, and there are two groups of order 64 which contain this H but no other abelian subgroup of order 16.

When H is of type (2, 1, 1) the Sylow subgroup of order 64 in its group of isomorphisms is the same as before, but this group of isomorphisms contains more sets of conjugate operators whose orders are powers of 2 than the preceding one. In particular, there are now three sets of conjugate cyclic subgroups of order 4 but only one of these involves operators of order 2 which are not commutative with half the operators of H. Hence there is only one

group of order 64 which contains this H and transforms it according to a cyclic group of order 4. To prove that there is also only one set of conjugate non-cyclic subgroups of order 4 which give rise to a group of order 64 which satisfies the given conditions, it may be noted that when the group of isomorphisms of this H is represented as a subgroup of order 192 in the holomorph of the regular group of order 8 and of type $(1, 1, 1)$, the subgroup of order 24 composed of all the substitutions which omit one letter of this group of order 192 is transitive on six letters. It contains two sets of conjugate non-cyclic subgroups of order 4. These are of degree 4 and 6 respectively. As only the latter can be used according to the considerations noted above it results that there are only two groups of order 64 which involve the abelian group of type $(2, 1, 1)$ but no other abelian group of order 16.

If H is of type $(2, 2)$ the Sylow subgroup whose order is a power of 2 in its group of isomorphisms is of order 32 and contains 19 operators of order 2 and 12 operators of order 4. It can be represented as a transitive group of degree 8 involving the direct product of two regular non-cyclic groups of order four. One of these corresponds to a cyclic commutator subgroup of order 4 and hence it requires no consideration here. The other two correspond to commutator subgroups of order 8. In the square of one of these all the operators of H correspond to their inverses while only half of them correspond to their inverses in the square of the other. As one group corresponds to each of these automorphisms there are two groups of order 64 which transform this H according to a cyclic group of order 4 and contain only one abelian subgroup of order 16.

When H is transformed according to a non-cyclic group of order 4 the commutator subgroups arising from its operators of order 2 cannot all be non-cyclic. Hence two of them must be cyclic groups of order 4 while the third is a non-cyclic group of this order. There are two such non-conjugate subgroups of order 4 in the group of isomorphisms of H. In one of these the operator of order 2 which gives rise to a non-cyclic commutator subgroup transforms every operator of H into its inverse while in the other it transforms only half of these operators into their inverses. In the former case there are two groups of order 64 while there is only one such group in the latter case. Hence there are five groups of order 64 which involve this H but no other abelian subgroup of order 16. Two of these transform it according to a cyclic group of order 4 while the other three transform it according to a non-cyclic group of this order. The total number of groups of order 64 which contain only one abelian subgroup of order 16 but no abelian subgroup of order 32 is therefore 9.

8. *Groups containing five abelian subgroups of order* 16 *and are generated by every pair of them.* When G contains at least two abelian subgroups of order 16 and is generated by every pair of its abelian subgroups of this order then H has a subgroup of order 4 in common with every other abelian subgroup of order 16. Let H_1 be one such subgroup and let K represent the cross-cut of H and H_1. It is easy to see that K is non-cyclic and that every operator of G which is not also in K has four conjugates under G. The central quotient group G/K involves only operators of order 2 and hence it is abelian. Every operator of G that is not also in K appears in one and only one abelian subgroup of order 16 and hence G must involve five invariant abelian subgroups of this order, and it is generated by every pair of these subgroups. Hence we have established the following theorem: *If a group of order* 64 *contains no abelian subgroup of order* 32 *but involves at least two abelian subgroups of order* 16 *and is generated by every pair of such subgroups contained therein then it contains exactly five abelian subgroups of order* 16, *and each of these subgroups is invariant.*

Suppose that at least one of the two abelian subgroups of order 16 contained in G is of type (1, 1, 1, 1). Exactly one-fourth of the remaining operators of G must be of order 2 since every operator of order 4 in G is transformed into its inverse by one-fourth of the operators of this subgroup. It therefore results that when G contains one abelian subgroup of type (1, 1, 1, 1) it also contains an invariant abelian subgroup H of type (2, 2). If an operator of the abelian subgroup of type (1, 1, 1, 1) transforms into their inverses all the operators of this H the remaining operators of the former subgroup must transform H in two ways determined by the two possible automorphisms of order 3 of the four group. On the other hand if half of the operators of H are transformed into their inverses by an operator of the said group of type (1, 1, 1, 1) the other two transformations under this subgroup are again completely determined, and hence there are two possible such groups of order 64 which involve an abelian subgroup of order 16 and of type (1, 1, 1, 1).

It remains to determine the groups in which the five abelian subgroups of order 64 are of the types (2, 2), (2, 1, 1). There is only one such group in which each of the five abelian subgroups is of type (2, 2) and there is no such group in which each of these subgroups is of type (2, 1, 1). Hence it results that *every group of order* 64 *which involves five abelian subgroups of order* 16 *but no subgroup of order* 32 *which is either abelian or has a central of order* 8 *contains at least one abelian subgroup of type* (2, 2). There is one such group which involves four abelian subgroups of type (2, 1, 1) and hence twenty operators of order 2, and there is one which contains three

abelian groups of type $(2, 2)$ and two of type $(2, 1, 1)$, and hence twelve operators of order 2. The total number of groups of order 64 which contain no abelian subgroup of order 32 but five abelian subgroups of order 16, and are generated by every pair of these subgroups is therefore 5. It remains to determine the groups of order 64 which involve a non-abelian subgroup of index 2 containing three abelian subgroups of order 16.

9. *Groups of order* 64 *arising from three-fourths automorphisms.* In what follows the symbol H will be used to represent a non-abelian subgroup of index 2 which contains three abelian subgroups of order 16. It is known that a characteristic property of such a group is that it admits three automorphisms in which exactly three-fourths of the operators correspond to their inverses. Such automorphisms have been called *three-fourths automorphisms,** and in the present section we aim to determine all the groups of order 64 which arise from a three-fourths automorphism of H but do not involve an abelian subgroup of order 32. Since one of the products of a three-fourths automorphism and the group of inner automorphisms is a characteristic operator in the group of automorphisms and the operators of the central correspond to their inverses under this automorphism, it results that the multiplier s_0 in the general theory of constructing all the groups which contain a given group as an invariant subgroup of prime index † can be selected only from the different sets of conjugates of order 2 in the central of H. Hence the following theorem: *The number of the distinct groups which involve as a subgroup of index* 2 *a given group* H *and transform it according to a three-fourths automorphism is equal to one more than the number of the different sets of operators of order* 2 *in the central of* H *which are conjugate under the group of isomorphisms of* H.

A group resulting from this theorem cannot involve an abelian subgroup of index 2 unless the central of H is of type $(1, 1, 1 \cdots)$. Moreover, when one of the abelian subgroups of index 2 in H is of type $(1, 1, 1 \cdots)$ the resulting group must involve an abelian subgroup of index 2. It remains therefore to consider the case when each of the abelian subgroups of index 2 in H is of type $(2, 1, 1 \cdots)$. If the resulting group does not contain an abelian subgroup of index 2 the operators of order 4 in H must have three distinct squares and the commutator of order 2 is the product of these squares. When H is of order 16 its central must therefore involve three operators of order 2 which are squares, three others which are the product of two squares

* G. A. Miller, *Proceedings of the National Academy of Sciences*, Vol. 15 (1929), p. 269.

† *Ibid.*, Vol. 14 (1928), p. 819.

and a commutator of order 2. Hence *there are four groups of order* 64 *which arise from an* H *whose central is of type* $(1, 1, 1)$ *by means of a three-fourths automorphism and do not involve an abelian subgroup of order* 32. It results directly from the theorem noted at the close of the preceding paragraph that when the central of H is a cyclic group of even order there are exactly two groups which arise from a three-fourths automorphism of H. Hence there are four such groups of order 64, two when H involves an operator of order 16 and two more when H involves three abelian subgroups of type $(3, 1)$.

When H involves an abelian subgroup of type $(3, 1)$ but its central is non-cyclic there are two such H's in which the commutator subgroup is generated by the square of an operator in the central and two others in which this is not the case. Each of the former gives rise to three groups while each of the latter gives rise to four groups, according to the general theorem noted above. Hence the total number of groups which results from these four H's when they are transformed according to a three-fourths automorphism is 14. When H involves no operator of order 8 but contains an abelian subgroup of type $(2, 2)$ there are three groups of order 32 which may be used for H. In one of these the commutator subgroup is generated by an operator of order 4 contained in the central, and this gives rise to three groups of order 64 while each of the other two gives rise to four groups of this order but only seven of these are distinct. It remains to consider the case when each of the three abelian subgroups of H is of type $(2, 1, 1)$, and the central of H is of type $(2, 1)$. Three groups of order 64 arise from this H, and hence the total number of groups of order 64 which arise from three-fourths automorphisms and involve no abelian subgroup of order 32 is 35.

10. *Groups of order* 64 *arising from other automorphisms.* It remains to consider the groups of order 64 which involve a subgroup H of index 2 admitting a three-fourths automorphism but are not transformed according to such an automorphism under G and do not contain any other subgroup of this index which is either abelian or thus transformed. When H contains a cyclic subgroup of order 16 all the operators of H may be transformed into their fifth powers under G and there is one such group and there is also one such group when the operators of one cyclic subgroup of order 16 are transformed into their third powers while those of the other are transformed into their eleventh powers. There are four sets of four operators with respect to the group of inner automorphisms which interchange the two cyclic groups of order 16 in H. In one of these sets there is an operator which is commutative with all the operators of an abelian subgroup of order 16 in H and hence only three of these sets require consideration here. The one which

involves only operators of order 4 gives rise to two groups while each of the others gives rise to only one group. There are therefore six additional groups when H contains an operator of order 16.

When H involves three abelian subgroups of type $(3, 1)$ and its central is the cyclic group of order 8 there is one automorphism of order 2 which transforms the operators of highest order in the central and in another cyclic group of order 8 into their fifth powers but is commutative with the remaining eight operators of this order. There is one group which corresponds to this automorphism. Under the Sylow subgroup of order 2^m in the group of isomorphisms of H these remaining operators of order 8 are transformed according to a transitive group of degree 8 and order 16 and hence this Sylow subgroup is the direct product of this group of order 16 and a group of order 2. There is clearly only one group which transforms H according to the product of a three-fourths automorphism and the given invariant operator of order 2. It remains to consider the cases when two cyclic groups of order 8 are transformed into each other. Four of these operators would give rise to an abelian subgroup of order 32 and hence they do not require consideration here. The product of one of these and the said invariant operator of order 2 gives rise to one group while the product of this and an operator which transforms H into a three-fourths automorphism gives rise to two groups. As the remaining product gives rise to only one group there are six groups which arise from this H.

There are four other H's which involve an abelian subgroup of type $(3, 1)$. When the commutator subgroup of H is generated by an operator of order 4 contained in such an H and one of the abelian subgroups of H is of type $(2, 1, 1)$ there are sixteen operators in the group of isomorphisms of H which transform each of its cyclic subgroups of order 8 into itself and these four subgroups are transformed according to the octic group under the group of isomorphisms. The said 16 operators are generated by a three-fourths automorphism and the 8 operators which leave invariant separately each operator in a subgroup of index 2 contained in H. These 8 operators do not give rise to a new group. Hence we need to consider only one of the operators which transform into itself each of the four cyclic groups of order 8 contained in H and this gives rise to only one group.

There are eight operators which interchange two of the four cyclic subgroups of order 8 which do not give rise to an additional group. The remaining eight give rise to six groups,—one set of four giving rise to four groups while the other gives rise to only two groups. When the four cyclic subgroups of order 8 are interchanged according to the invariant operator of order 2 in the octic group there is only one set of four operators which

do not give rise to an additional group while there are eight groups which correspond to the remaining three such sets. When these subgroups are transformed according to a non-invariant substitution of order 2 and degree 4 in the octic group there is again a set of four operators which does not give rise to an additional group, while the other three sets of four operators give rise to seven groups. As there is obviously no group which transforms the four cyclic groups of order 4 in H according to an operator of order 4 in the given octic group this H gives rise to 22 additional groups.

We proceed to consider the possible groups which result from the H which has the same commutator subgroup as the preceding H but one of the abelian subgroups is of type $(2, 2)$ instead of type $(2, 1, 1)$. The four cyclic subgroups of order 4 in this H are again transformed according to the octic group under its group of isomorphisms but the number of subgroups of index 2 in this H is 3 while there are 7 such subgroups in the preceding H. Hence there are only eight operators in the group of isomorphisms of the present H which transform into itself each of its cyclic subgroups of order 8. Hence there are 3 groups of order 64 which contain this H and interchange only two of its cyclic subgroups of order 8 and there are 3 more which transform the four cyclic subgroups in pairs according to the invariant operator of order 2 in the octic group. As there are four groups when these four subgroups are transformed according to a non-invariant operator in the octic group, the number of the groups which result from this H is 10.

When H has as a commutator subgroup a non-characteristic subgroup of order 2 its group of isomorphisms is again of order 64 and there are four groups which interchange only two of its cyclic subgroups of order 8. There are also four groups which interchange these cyclic subgroups according to the invariant operator of order 2 in the octic group. As there are also four groups when the four subgroups are permuted according to a non-invariant operator of order 2 in the octic group the number of groups which arise from this H is 12. There is only one other possible H which involves operators of order 8 and there are again only 8 operators in its group of isomorphisms which transform each of the four cyclic subgroups of order 8 in H into itself. Just as in the preceding case there result 8 groups when only two cyclic subgroups of order 8 in H are interchanged or when these cyclic subgroups are interchanged according to the invariant operator of order 2 in the octic group. When the four cyclic subgroups of order 8 are interchanged according to a non-invariant operator of order 2 in the octic group there are also four groups, and hence the number of groups which result from this H is also 12. The total number of groups which involve an H containing operators of order 8 but none of order 16 is therefore 62.

It remains to consider the groups which do not involve an H containing operators of order 8. There is one such H which involves three abelian subgroups of type $(2, 2)$ and its group of isomorphisms is of order 32. Half of these transform into itself all the operators of the central of H and include the group of inner isomorphisms but no three-fourths isomorphism. These give rise to groups which contain an abelian subgroup of index 2 and hence they need no consideration here. Hence there are only four groups resulting from this H. When two of the abelian subgroups are of type $(2, 2)$ and the third is of type $(2, 1, 1)$ there are six groups which arise from the automorphisms which are similar to those of the preceding H. The present H has, however, 32 additional automorphisms which transform into each other two operators of order 2 in its central. There are twelve groups which correspond to these automorphisms. When only one of the abelian subgroups of order 16 in H is of type $(2, 2)$ its group of isomorphisms is of order 32 and the three operators of order 2 in the central of H are characteristic. Eight groups result from this H. When the three abelian subgroups of order 16 are of type $(2, 1, 1)$ and the central is of type $(2, 1)$ the group of isomorphisms of H contains a Sylow subgroup of order 64, and H involves 15 subgroups of index 2. The groups arising from these isomorphisms are covered by the following theorem: *There are four and only four groups of order* 64 *which satisfy the following conditions: They involve no operator whose order exceeds* 4 *and their operators of order* 4 *have a common square which generates their commutator subgroup.* Two of these four groups arise from three-fourths automorphisms while the other two do not have this property. The latter contain the H under consideration.

When half of the operators of H not including the central are transformed into themselves and the rest into themselves multiplied by a non-characteristic operator of order 2 there are two groups. When all of the abelian subgroups of index 2 in H are of type $(2, 1, 1)$ and the central is of type $(1, 1, 1)$ there are two groups when the operators of order 4 have a common square and eight groups when the squares are not the same. When one of the abelian subgroups is of type $(1, 1, 1, 1)$ and the other two are of type $(2, 1, 1)$ there are four groups. Finally, when two of the abelian subgroups of order 16 are of type $(1, 1, 1, 1)$ there are no additional groups. Hence the total number of groups of order 64 which do not contain a subgroup of order 32 involving an abelian subgroup of order 16 which contains operators of order 8 is 48 and the total number of groups of order 64 is 294. The number of those which do not involve an abelian subgroup of index 2 is 165.

A General System of Ordinary Differential Equations of the First Order.*

By Eugene Feenberg.

Introduction to Part 1.

The object of this section is to determine necessary and sufficient conditions that a positive valued scalar function $H[w_1(x), w_2(x), \cdots, w_j(x) \cdots]$, satisfy the inequality

$$1.1 \quad H\left[\int_a^x w_1(t)dt, \int_a^x w_2(t)dt, \cdots\right] \leqq \int_a^x H[w_1(t), w_2(t), \cdots]dt.$$

The following notation and definitions are used thruout the paper: Vectors are represented by Clarendon type, scalars and components of vectors by light face type. $\boldsymbol{w} \equiv (w_1, w_2, \cdots)$, an ordered set of numbers is a vector. $\boldsymbol{w}(k)$ is the vector obtained from $\boldsymbol{w}$ by taking

$$w_j(k) = w_j, \qquad j \leqq k,$$
$$w_j(k) = 0, \qquad j > k.$$

A vector is said to have a certain property when all the components of the vector have that property. A scalar function of a vector, $F(\boldsymbol{w})$, will be written as $F(w_j)$ when all the components of $\boldsymbol{w}$ except w_j are zero. A vector is on the "extended" range $\boldsymbol{D}$, [$\boldsymbol{D}$ a sequence of positive numbers, $(D_1, D_2, \cdots)$], if there exists a positive number e such that $-e\boldsymbol{D}_j < w_j < eD_j$, $(j = 1, 2, \cdots)$.

Part 1.

Consider an arbitrary range $\boldsymbol{D}$ and a positive valued scalar function, $H(\boldsymbol{w})$, defined for all values of $\boldsymbol{w}$ on the extended range $\boldsymbol{D}$. In the following theorems the symbols $\boldsymbol{w}, \boldsymbol{w}', \boldsymbol{w}''$, $\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{z}''$ represent vectors on the extended range $\boldsymbol{D}$.

Theorem I. *Necessary conditions that* $H(\boldsymbol{w})$ *satisfy the inequality* $H\left[\int_a^x \boldsymbol{w}(t)dt\right] \leqq \int_a^x H[\boldsymbol{w}(t)]\,dt$, *where* $\boldsymbol{w}(x)$ *is a summable vector,* $H[\boldsymbol{w}(x)]$ *is summable, and* $\int_a^x \boldsymbol{w}(t)dt$ *is on the extended range* $\boldsymbol{D}$, $(a \leqq x \leqq b)$, *are*

* Presented to the American Mathematical Society, September 7, 1928.

$$H(c\boldsymbol{w}) = cH(\boldsymbol{w}), \quad c \geqq 0, \qquad H(\boldsymbol{w}' + \boldsymbol{w}'') \leqq H(\boldsymbol{w}') + H(\boldsymbol{w}'').$$

Proof. Take $\boldsymbol{w}(x) = \boldsymbol{w}$, ($\boldsymbol{w}$ a constant vector). Then $H(\int_0^x \boldsymbol{w}dt) = H(x\boldsymbol{w})$ and $\int_0^x H(\boldsymbol{w})dt = xH(\boldsymbol{w})$. By (1.1)

1.2 $$H(c\boldsymbol{w}) \leqq cH(\boldsymbol{w}), \quad c \geqq 0.$$

c and $\boldsymbol{w}$ are arbitrary; replacing c by $1/c$ and $\boldsymbol{w}$ by $c\boldsymbol{w}$ $H(\boldsymbol{w}) \leqq 1/cH(c\boldsymbol{w})$.

1.3 $$H(c\boldsymbol{w}) \geqq cH(\boldsymbol{w}).$$

Hence

1.4 $$H(c\boldsymbol{w}) = cH(\boldsymbol{w}), \quad c \geqq 0.$$

Take $\boldsymbol{w}(x) = \boldsymbol{w}'$, $0 \leqq x \leqq 1$, $\boldsymbol{w}(x) = \boldsymbol{w}''$, $1 < x \leqq 2$. By (1.1)

$$H(\int_0^1 w'dt + \int_1^2 \boldsymbol{w}''dt) \leqq \int_0^1 H(\boldsymbol{w}')dt + \int_1^2 H(\boldsymbol{w}'')dt.$$

Hence

1.5 $$H(\boldsymbol{w}' + \boldsymbol{w}'') \leqq H(\boldsymbol{w}') + H(\boldsymbol{w}'').$$

It will be shown that conditions (1.4) and (1.5) are sufficient to prove the inequality (1.1) for the case of a finite number of variables.

THEOREM II. (1.4) *and* (1.5) *are sufficient conditions that* $H[\boldsymbol{w}(k)]$ *be continuous in* $\boldsymbol{w}(k)$, $(k = 1, 2, \cdots)$.

Proof. Define I_j as the vector with every component zero except the j-th component which is identically one. By (1.4)

$$H(c\boldsymbol{w}) = cH(\boldsymbol{w}), \quad H[c(-\boldsymbol{w})] = cH(-\boldsymbol{w}), \qquad c \geqq 0.$$

Take $\boldsymbol{w} = I_j$, $c = |v_j|$, v_j an arbitrary number. Then

$$H(c\boldsymbol{w}) = H(|v_j| I_j) = |v_j| H(I_j).$$

Take $\boldsymbol{w} = -I_j$, $c = |v_j|$.

$$H(c\boldsymbol{w}) = H(-|v_j| I_j) = |v_j| H(-I_j).$$

Hence

1.6 $$H(w_j) \leqq |w_j| [H(I_j) + H(-I_j)],$$

Applying (1.5) k times

1.7 $$H[\boldsymbol{w}(k)] \leqq \sum_{j=1}^{k} H(w_j) \leqq \sum_{j=1}^{k} |w_j| [H(I_j) + H(-I_j)], \qquad (k = 1, 2, \cdots).$$

Thus since $H(0)=0$ and $\underset{w(k)\to 0}{\text{Limit}}\, H[\boldsymbol{w}(k)]=0$, $H[\boldsymbol{w}(k)]$ is continuous at $\boldsymbol{w}(k)=0$, $(k=1,2,\cdots)$, Let $\boldsymbol{w}'+\boldsymbol{w}''=\boldsymbol{z}'$, $\boldsymbol{w}''=\boldsymbol{z}''$. By (1.5)

1.8 $$H(\boldsymbol{z}'-\boldsymbol{z}'')\geqq H(\boldsymbol{z}')-H(\boldsymbol{z}'')$$

and similarly

1.9 $$H(\boldsymbol{z}''-\boldsymbol{z}')\geqq H(\boldsymbol{z}'')-H(\boldsymbol{z}').$$

Take $\boldsymbol{z}'=\boldsymbol{z}'(k)$, $\boldsymbol{z}''=\boldsymbol{z}''(k)$. Then

$$\underset{z''\to z'}{\text{Limit}}\, H[\boldsymbol{z}'(k)-\boldsymbol{z}''(k)]=\underset{z''\to z'}{\text{Limit}}\, H[\boldsymbol{z}''(k)-\boldsymbol{z}'(k)]=0,$$
$$\underset{z''\to z'}{\text{Limit}}\, \{H[\boldsymbol{z}'(k)]-H[\boldsymbol{z}''(k)]\}\leqq 0,$$
$$\underset{z''\to z'}{\text{Limit}}\, \{H[\boldsymbol{z}''(k)]-H[\boldsymbol{z}'(k)]\}\leqq 0.$$

Hence $\underset{z''\to z'}{\text{Limit}}\, H[\boldsymbol{z}''(k)]=H[\boldsymbol{z}'(k)]$ and $H[\boldsymbol{w}(k)]$ is continuous in $\boldsymbol{w}(k)$, $(k=1,2,\cdots)$.

Take $(\boldsymbol{w}^{(1)},\boldsymbol{w}^{(2)},\cdots)$ a sequence of vectors such that each vector and $\sum_{j=1}^{\infty}\boldsymbol{w}^{(j)}$ are on the extended range $\boldsymbol{D}$ and $\sum_{j=1}^{\infty}H(\boldsymbol{w}^{(j)})$ converges. That such a sequence exists is readily shown from the inequality (1.7). Define

$$\boldsymbol{w}(x)=2^j\boldsymbol{w}^{(j)},\quad 2^{-j}<x\leqq 2^{-j+1},$$
$$(j=1,2,\cdots).$$

Then by (1.1)

1.10 $$H[\sum_{n+1}^{\infty}\boldsymbol{w}^{(j)}]=H[\int_0^{2^{-n}}\boldsymbol{w}(t)\,dt]\leqq\int_0^{2^{-n}}H[\boldsymbol{w}(t)]\,dt=\sum_{n+1}^{\infty}H(\boldsymbol{w}^{(j)}).$$

Hence

1.11 $$\underset{n\to\infty}{\text{Limit}}\, H[\sum_{j=n}^{\infty}\boldsymbol{w}^{(j)}]=0.$$

In particular if $\boldsymbol{w}$ is a vector such that $\sum_{j=n}^{\infty}[H(w_j)+H(-w_j)]$ exists, then $\underset{k\to\infty}{\text{Limit}}\, H[\boldsymbol{w}-\boldsymbol{w}(k)]=\underset{k\to\infty}{\text{Limit}}\, H[\boldsymbol{w}(k)-\boldsymbol{w}]=0$, and $\underset{k\to\infty}{\text{Limit}}\, H[\boldsymbol{w}(k)]=H(\boldsymbol{w})$.

To prove the sufficiency of conditions (1.4) and (1.5) we make use of a corollary to Valleé Poussin's * integral theorem. For clearness the theorem is restated here in a restricted form.

Given a sequence of functions $f_n(x)\geqq 0$, $(n=1,2,\cdots)$, and a function $f(x)\geqq 0$ such that on the interval (a,b)

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* "Sur L'Intégrale de Lebesgue," *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 435.

c_1. $\underset{n\to\infty}{\text{Limit}}\, f_n(x) = f(x)$ "a. e.",

c_2. $f_n(x)$ is summable, $(n = 1, 2, \cdots)$,

c_3. $\int_a^x f_n(t)\,dt$ is absolutely continuous uniformly with respect to n.

Conclusion: c_1, c_2, c_3 are necessary and sufficient for c_4.

c_4. $f(x)$ is summable and $\underset{n\to\infty}{\text{Limit}} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt$. Divide the interval (a, b) into n subintervals taking $a = t_0 \leqq t_{1n} \cdots \leqq t_{nn} = b$ as the points of division. A horizontal function of index n, $h_n(x)$, is defined to be a function such that $h_n(x) = h_{in}$, (h_{in} a constant), on $t_{in} \leqq x < t_{i+1n}$.* If $F(x)$ is a summable function on (a, b) there exist for any method of fine subdivision, a sequence of horizontal functions, $h_n(x)$, such that $\underset{n\to\infty}{\text{Limit}}\, h_n(x) = F(x)$ "a. e." on (a, b), $\underset{n\to\infty}{\text{Limit}} \int_a^x h_n(t)\,dt$ and $\underset{n\to\infty}{\text{Limit}} \int_a^x |\, h_n(t)\,|\, dt$ exist independent of the method of fine subdivision. Take $F(x)$ a non-negative summable function and $h_n(x) \geqq 0$. Then the corollary states that $\int_a^x h_n(t)\,dt$ is absolutely continuous in x uniformly with respect to n. To prove this identify $h_n(x)$ with $f_n(x)$ and $F(x)$ with $f(x)$. c_1, c_2, c_4 of the theorem stated above are satisfied. Therefore c_3 holds proving the corollary.

THEOREM III. *Given the conditions* (1. 4), (1. 5) *and a vector,* $g(x, k)$, *summable on* (a, b), *then*

$$H[\int_a^x \boldsymbol{g}(t, k)\,dt] \leqq \int_a^x H[\boldsymbol{g}(t, k)]\,dt, \quad (a \leqq x \leqq b), \quad (k = 1, 2, \cdots).$$

Proof. There exists a sequence of horizontal vectors, $h^{(n)}(x, k)$, such that $\underset{n\to\infty}{\text{Limit}}\, h_j^{(n)}(x) = g_j(x)$ "a. e." on (a, b), $\underset{n\to\infty}{\text{Limit}} \int_a^x h_j^{(n)}(t)\,dt = \int_a^x g_j(t)\,dt$ and $\underset{n\to\infty}{\text{Limit}} \int_a^x |\, h_j^{(n)}(t)\,|\, dt = \int_a^x |\, g_j(t)\,|\, dt$, $(j = 1, 2, \cdots)$. By theorem II.

$$\underset{n\to\infty}{\text{Limit}}\, H[\boldsymbol{h}^{(n)}(x, k)] = H[\boldsymbol{g}(x, k)] \text{ "a. e." on } (a, b).$$

$$\underset{n\to\infty}{\text{Limit}}\, H[\sum_{i=1}^{i_{xn}} \boldsymbol{h}^{in}(k)\Delta_{in}] = \underset{n\to\infty}{\text{Limit}}\, H[\int_a^x \boldsymbol{h}^{in}(t, k)\,dt] = H[\int_a^x \boldsymbol{g}(t, k)\,dt].\dagger$$

By conditions (1. 4) and (1. 5)

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* H. J. Ettlinger, "On Multiple Iterated Integrals," *American Journal of Mathematics*, Vol. 48 (1926), p. 215.

† i_{xn} is for each value of n the subscript of the last division point preceding the point x, i. e., $t_{i_{xn}} < x \leqq t_{i_{xn}+1}$.

$$H[\sum_{i=1}^{i_{sn}} \boldsymbol{h}^{in}(k)\Delta_{in}] \leqq \sum_{i=1}^{i_{sn}} H[\boldsymbol{h}^{in}(k)]\Delta_{in}.$$

1.12
$$\int_{x_1}^{x_2} H[\boldsymbol{h}^{(n)}(t,k)]dt \leqq \int_{x_1}^{x_2} \sum_{j=1}^{k} H[h_j^{(n)}(t)]dt$$
$$\leqq \sum_{j=1}^{k} [H(I_j)+H(-I_j)] \int_{x_1}^{x_2} |\, h_j^{(n)}(t)\,|\, dt.$$

Therefore

$$\operatorname*{Limit}_{n\to\infty} \{\sum_{i=1}^{i_{sn}} H[h^{in}(k)]\Delta_{in} - \int_a^x H[\boldsymbol{h}^{(n)}(t,k)]dt\} = 0, \int_a^x H[\boldsymbol{h}^{(n)}(t,k)]dt$$

is absolutely continuous in x uniformly with respect to n and

$$\operatorname*{Limit}_{n\to\infty} \sum_{i=1}^{i_{sn}} H[\boldsymbol{h}^{in}(k)]\Delta_{in} = \int_a^x H[\boldsymbol{g}(t,k)]dt.$$

Hence

1.13
$$H[\int_a^x \boldsymbol{g}(t,k)dt] \leqq \int_a^x H[\boldsymbol{g}(t,k)]dt,\ (k=1,2,\cdots).$$

THEOREM IV. *Given conditions* (1.4), (1.5) and a summable vector, $\boldsymbol{g}(x)$, *such that* $H[\boldsymbol{g}(x)]$ *is finite and* $\operatorname*{Limit}_{k\to\infty} H[\boldsymbol{g}(x,k)] = H[\boldsymbol{g}(x)]$ "a. e." *on* (a,b), $\operatorname*{Limit}_{k\to\infty} H[\int_a^x \boldsymbol{g}(t,k)dt] = H[\int_a^x \boldsymbol{g}(t)dt]$, $\int_a^x H[\boldsymbol{g}(t,k)]dt$ *is absolutely continuous in* x *uniformly with respect to* k. *Then*

$$H[\int_a^x \boldsymbol{g}(t)dt] \leqq \int_a^x H[\boldsymbol{g}(t)]dt.$$

Proof. By theorem III.

$$\operatorname*{Limit}_{k\to\infty} H[\int_a^x \boldsymbol{g}(t,k)dt] \leqq \operatorname*{Limit}_{k\to\infty} \int_a^x H[\boldsymbol{g}(t,k)]dt.$$

By Valleé Poussin's theorem $H[\boldsymbol{g}(x)]$ is summable and

$$\operatorname*{Limit}_{k\to\infty} \int_a^x H[\boldsymbol{g}(t,k)]dt = \int_a^x H[\boldsymbol{g}(t)]dt.$$

Therefore

$$H[\int_a^x \boldsymbol{g}(t)dt] \leqq \int_a^x H[\boldsymbol{g}(t)]dt.$$

. . . . . .

The function $H(\boldsymbol{w})=(\sum_{j=1}^{\infty} |\, w_j\,|^p)^{1/p}$, $p \geqq 1$, obviously has the property (1.4). That (1.5) is a property of this function has been shown by Friedrich Riesz.*

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* "Untersuchungen über Systeme Integrierbarer Funktionen," *Mathematische Annalen*, Vol. 68 (1910), p. 455.

Using Riesz's result the function

$$H(\boldsymbol{w}) = [\sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} | w_j |^{q_i})^{p/q_i} c_i]^{1/p}, \qquad (p \geqq 1),\ (q_i \geqq 1),\ (c_i \geqq 0),$$

is readily found to satisfy conditions (1.4) and (1.5).

If $H(w)$ is an even function, [i.e., $H(\boldsymbol{w}) = H(-\boldsymbol{w})$], the properties

1.14 $$H(\boldsymbol{w} - \boldsymbol{w}') \geqq | H(\boldsymbol{w}) - H(\boldsymbol{w}') |,$$

1.15 $$H(\boldsymbol{w} - \boldsymbol{w}') \geqq | H(\boldsymbol{w} + \boldsymbol{z}) - H(\boldsymbol{w}' + \boldsymbol{z}) |.$$

follow immediately from (1.8) and (1.9).

Introduction to Part 2.

Systems of ordinary differential and difference equations in a countable number of variables are treated in Part 2. Such systems have been studied by a number of authors.* It is the purpose of this paper to apply the results of the preceding section to the establishment of certain general existence theorems.

The set of hypotheses Q are as follows:

I. $\boldsymbol{a} \equiv (a_1, a_2, \cdots)$ is a constant vector. $\boldsymbol{D}^*(x)$ is a positive vector function bounded by a constant positive vector $\boldsymbol{R}$, i.e., $0 \leqq D_j^*(x) \leqq r_j$. The vector arguments $\boldsymbol{w}(x)$, $\boldsymbol{w}'(x)$, $\boldsymbol{w}^{(l)}(x)$, $\boldsymbol{w}'^{(l)}(x)$, $(l = 1, 2, \cdots)$, used in the following hypotheses are any set of vectors such that for any vector, $\boldsymbol{w}(x)$, of the set, $| w_j(x) - a_j | \leqq D_j^*(x) \leqq r_j$, $(j = 1, 2, \cdots)$. The properties stated are with respect to the interval $0 \leqq x \leqq b$.

II. $\boldsymbol{f}[x, \boldsymbol{w}(x)] \equiv \{f_1[x, \boldsymbol{w}(x)], f_2[x, \boldsymbol{w}(x)], \cdots\}$ is a summable vector such that $| \int_a^x f_j[t, \boldsymbol{w}(t)]\,dt | \leqq D_j^*(x)$, $(j = 1, 2, \cdots)$.

III. $[H_1(\boldsymbol{w}), H_2(\boldsymbol{w}), \cdots]$ is a sequence of positive valued scalar functions such that on the range $\boldsymbol{D}$: $D_j = 2(| a_j | + r_j)$, $(j = 1, 2, \cdots)$,

a. $H_j(\boldsymbol{w}) \leqq M_j$, M_j a positive number,

b. $H_j[\int_0^x \boldsymbol{g}(t)\,dt] \leqq \int_0^x H_j[\boldsymbol{g}(t)]\,dt$ when $H_j[\boldsymbol{g}(x)]$, is summable and $\int_0^x \boldsymbol{g}(t)\,dt$ is on $\boldsymbol{D}$, $(j = 1, 2, \cdots)$.

IV. There exist sequences of positive valued summable functions:

2.1 $$\{G_1(x, \boldsymbol{w}, \boldsymbol{w}'), G_2(x, \boldsymbol{w}, \boldsymbol{w}'), \cdots\},$$

* W. L. Hart, "Differential Equations and Implicit Functions in Infinitely Many Variables," *Transactions of the American Mathematical Society*, Vol. 18 (1917).

2.2 $\{K_i(x), K_2(x), \cdots\}$, such that

2.3 $\sum_{l=0}^{\infty} 1/l![\int_0^b K_j(t)dt]^l \int_0^a G_j[t, \boldsymbol{w}^{(l)}(t), \boldsymbol{w}'^{(l)}(t)]dt$ *exists.*

V. Finally hypotheses III and IV are such that:

2.4 $$|\int_{x_1}^{x_2}\{f_j[t, \boldsymbol{w}(t)] - f_j[t, \boldsymbol{w}'(t)]\}dt| \leqq \int_{x_1}^{x_2} G_j[t, \boldsymbol{w}(t), \boldsymbol{w}'(t)]H_j[\boldsymbol{w}(t) - \boldsymbol{w}'(t)]dt,$$

2.5 $$H_j[\int_{x_1}^{x_2}\{\boldsymbol{f}[t, \boldsymbol{w}(t)] - \boldsymbol{f}[t, \boldsymbol{w}'(t)]\}dt] \leqq \int_{x_1}^{x_2} K_j(t)H_j[\boldsymbol{w}(t) - \boldsymbol{w}'(t)]dt,$$
$$0 \leqq x_1 \leqq x_2 \leqq b, \qquad (j = 1, 2, \cdots).$$

Consider the system $\boldsymbol{y}(x) = \int_0^x \boldsymbol{f}[t, \boldsymbol{y}(t)]dt + \boldsymbol{a}$. A vector absolutely continuous in x and satisfying the system is a solution. It is shown that under the set of hypotheses Q the system has one and only one solution on $\boldsymbol{D}^*(x)$.

A difference system is derived from the differential system and with a slight modification of the hypotheses Q the existence of a unique solution is established. Let the interval $(0, b)$ be divided into n subintervals by the points $0 = t_{0n} \leqq t_{1n} \cdots \leqq t_{nn} = b$. $w_n(x) = w_n(i)$ is defined to be a vector, the components of which are horizontal functions constant on the subintervals $t_{in} \leqq x < t_{i+1,n}$, $(i = 1, 2, \cdots, n)$. Defining

$$\boldsymbol{B}_n[i-1, \boldsymbol{w}_n(i-1)] \equiv \int_{t_{i-1,n}}^{t_{in}} \boldsymbol{f}[t, \boldsymbol{w}_n(i-1)]dt,$$

the difference system is

$$\boldsymbol{y}_n(i) = \sum_{r=0}^{i-1} \boldsymbol{B}_n[r, \boldsymbol{y}_n(r)] + \boldsymbol{a}.$$

With the additional hypotheses: $H_j[\boldsymbol{f}(x, \boldsymbol{a})]$ summable, $G_j(x, \boldsymbol{w}, \boldsymbol{w}') = G_j(x)$, $G_j(x)$ a summable function, $(j = 1, 2, \cdots)$, the solution is shown to pass over in the limit as n becomes infinite, for any method of fine subdivision, to the solution of the corresponding differential system.

Part 2.

Theorem V. *Given the set of hypotheses Q then the system*

$$\boldsymbol{y}(x) = \int_0^x \boldsymbol{f}[t, \boldsymbol{y}(t)]dt + \boldsymbol{a}$$

has one and only one solution on $\boldsymbol{D}^*(x)$, $0 \leqq x \leqq b$.

Proof. By the usual method of successive approximation take

$$
\begin{aligned}
&\boldsymbol{y}^{(0)}(x) = \boldsymbol{a}, \\
&\boldsymbol{y}^{(1)}(x) = \int_0^x \boldsymbol{f}[t, \boldsymbol{y}^{(0)}(t)]dt + \boldsymbol{a}, \\
&\boldsymbol{y}^{(2)}(x) = \int_0^x \boldsymbol{f}[t, \boldsymbol{y}^{(1)}(t)]dt + \boldsymbol{a}, \\
&\quad\cdot \quad\cdot \quad\cdot \quad\cdot \quad\cdot \quad\cdot \\
&\boldsymbol{y}^{(k)}(x) = \int_0^x \boldsymbol{f}[t, \boldsymbol{y}^{(k-1)}(t)]dt + \boldsymbol{a}, \qquad (k = 1, 2, \cdots).
\end{aligned} \tag{2.6}
$$

By hypotheses Q, II. this sequence exists for all values of k. Define

$$
\begin{aligned}
&\boldsymbol{U}^{(1)}(x) = \boldsymbol{y}^{(1)}(x), \\
&\boldsymbol{U}^{(2)}(x) = \boldsymbol{y}^{(2)}(x) - \boldsymbol{y}^{(1)}(x), \\
&\quad\cdot \quad\cdot \quad\cdot \quad\cdot \quad\cdot \quad\cdot \\
&\boldsymbol{U}^{(k)} = \boldsymbol{y}^{(k)}(x) - \boldsymbol{y}^{(k-1)}(x), \qquad (k = 1, 2, \cdots).
\end{aligned} \tag{2.7}
$$

Then

$$
\boldsymbol{y}^{(k)}(x) = \boldsymbol{y}^{(1)}(x) + \sum_{l=2}^{k} \boldsymbol{U}^{(l)}(x).
$$

By hypotheses Q, II and Q, line 2.4

$$
\begin{aligned}
&| U_j^{(1)}(x) | \leqq r_j + | a_j |, \\
&| U_j^{(k)}(x) | \leqq \int_0^x G_j[t, \boldsymbol{y}^{(k-1)}(t), \boldsymbol{y}^{(k-2)}(t)] H_j[t, \boldsymbol{U}^{(k-1)}(t)]dt, \ (k \geqq 2).
\end{aligned} \tag{2.8}
$$

By hypotheses Q, III and Q, line 2.5

$$
H_j[\boldsymbol{U}^{(k)}(x)] \leqq M_j, \ H_j[\boldsymbol{U}^{(k)}(x)] \leqq \int_0^x K_j(t) H_j[\boldsymbol{U}^{(k-1)}(t)]dt, \ (k \geqq 2). \tag{2.9}
$$

Hence by induction

$$
H_j[\boldsymbol{U}^{(k)}(x)] \leqq M_{j/(k-1)!} \left[\int_0^b K_j(t)dt \right]^{k-1}, \qquad (k = 1, 2, \cdots), \tag{2.10}
$$

$$
\begin{aligned}
&| U_j^{(k)}(x) | \\
&\leqq M_{j/(k-2)!} \left[\int_0^b K_j(t)dt \right]^{k-2} \int_0^b G_j[t, \boldsymbol{y}^{(k-1)}(t), \boldsymbol{y}^{(k-2)}(t)]dt, \ (k \geqq 2),
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
&\sum_{l=1}^{\infty} | U_j^{(l)}(x) | \\
&\leqq \sum_{l=0}^{\infty} M_{j/l!} \left[\int_0^b K_j(t)dt \right]^l \int_0^b G_j[t, \boldsymbol{y}^{(l)}(t), \boldsymbol{y}^{(l-1)}(t)]dt + | a_j | + y_j.
\end{aligned} \tag{2.12}
$$

Therefore $\underset{k\to\infty}{\text{Limit}}\ y_j^{(k)}(x) = \underset{k\to\infty}{\text{Limit}} \sum_{l=1}^{k} U_j^{(l)}(x)$ exists [$= y_j(x)$ say].

$$
\underset{k\to\infty}{\text{Limit}}\ y_j^{(k)}(x) = y_j(x) \tag{2.13}
$$

$\boldsymbol{D}^*(x)$ for any fixed value of x is a closed region. Therefore $\boldsymbol{y}(x)$ is on $\boldsymbol{D}^*(x)$. It remains to prove that $\boldsymbol{y}(x)$ is a solution of the system. By 2.6

$$\operatorname*{Limit}_{k\to\infty} y_j^{(k)}(x) = y_j(x) = \operatorname*{Limit}_{k\to\infty} \int_0^x f_j[t, \mathbf{y}^{(k-1)}(t)]dt + a_j.$$

$$\left| \int_0^x \{f_j[t, \mathbf{y}(t)] - f_j[t, \mathbf{y}^{(k)}(t)]\}dt \right|$$

$$\leqq \int_0^x G_j[t, \mathbf{y}(t), \mathbf{y}^{(k)}(t)] H_j[\mathbf{y}(t) - \mathbf{y}^{(k)}(t)]dt$$

$$= \int_0^x G_j[t, \mathbf{y}(t), \mathbf{y}^{(k)}(t)] H_j[\sum_{l=k+1}^{\infty} \boldsymbol{U}^{(l)}(t)]dt$$

$$\leqq \int_0^x G_j[t, \mathbf{y}(t), \mathbf{y}^{(k)}(t)] \sum_{l=k+1}^{\infty} H_j[\boldsymbol{U}^{(l)}(t)]dt$$

$$\leqq \int_0^x G_j[t, \mathbf{y}(t), \mathbf{y}^{(k)}(t)]dt \sum_{l=k+1}^{\infty} M_{j/(l-1)!} \left[\int_0^b K_j(t)dt\right]^{l-1}$$

$$\leqq M_j \{\left[\int_0^b K_j(t)dt\right]^k / k! \int_0^b G_j[t, \mathbf{y}(t), \mathbf{y}^{(k)}(t)]dt\} \exp\left[\int_0^b K_j(t)dt\right].$$

By *Q*, 2. 3 the term in brackets is the k-th term of a convergent series. Therefore

2. 14 $$\operatorname*{Limit}_{k\to\infty} \int_0^x f_j[t, \mathbf{y}^{(k)}(t)]dt = \int_0^x f_j[t, \mathbf{y}(t)]dt.$$

Hence

$$y_j(x) = \int_0^x f_j[t, \mathbf{y}(t)]dt + a_j, \qquad (j = 1, 2, \cdots).$$

$$\mathbf{y}(x) = \int_0^x \boldsymbol{f}[t, \mathbf{y}(t)]dt + \boldsymbol{a}$$ and a solution exists.

To prove uniqueness assume there exists another solution, $\mathbf{y}'(x)$, on $\boldsymbol{D}^*(x)$. Then

$$\mathbf{y}(x) - \mathbf{y}'(x) \equiv \int_0^x \{\boldsymbol{f}[t, \mathbf{y}(t)] - \boldsymbol{f}[t, \mathbf{y}'(t)]\}dt,$$

$$H_j[\mathbf{y}(x) - \mathbf{y}'(x)] \leqq \int_0^x K_j(t) H_j[\mathbf{y}(t) - \mathbf{y}'(t)]dt \leqq M_j \int_0^x K_j(t)dt.$$

It follows by induction that

$$H_j[\mathbf{y}(x) - \mathbf{y}'(x)] \leqq M_j / n! \left[\int_0^b K_j(t)dt\right]^n$$

where n is any positive integer. Hence $H_j[\mathbf{y}(x) - \mathbf{y}'(x)] \equiv 0$ and

$$|\, y_j(x) - y_j'(x) \,| \leqq \int_0^x G_j[t, \mathbf{y}(t), \mathbf{y}'(t)] H_j[\mathbf{y}(t) - \mathbf{y}'(t)]dt \equiv 0.$$

Therefore $\mathbf{y}'(x) = \mathbf{y}(x)$ and solution is unique.

The proof of the existence theorem goes thru, with slight changes in the method of proof, if we require that the hypotheses *Q*, II, III, IV, V are necessarily true only when the vector arguments $\boldsymbol{w}(x)$, $\boldsymbol{w}'(x)$, $\boldsymbol{w}^{(1)}(x)$,

$\boldsymbol{w}'^{(1)}(x)$ are on $\boldsymbol{D}^*(x)$ and are absolutely continuous on $(0, b)$ and add the hypothesis that the function defined by the series Q, 2.3 is absolutely continuous on $(0, b)$. Sufficient conditions that Q, 2.4 and Q, 2.5 follow from the other hypotheses of the set Q are

2.15 $\quad |f_j(x, \boldsymbol{w}) - f_j(x, \boldsymbol{w}')| \leqq G_j(x, \boldsymbol{w}, \boldsymbol{w}') H_j(\boldsymbol{w} - \boldsymbol{w}'),$

2.16 $\quad H_j[\boldsymbol{f}(x, \boldsymbol{w}) - \boldsymbol{f}(x, \boldsymbol{w}')] \leqq K_j(x) H_j(\boldsymbol{w} - \boldsymbol{w}').$

THEOREM VI. *Given the set of hypotheses Q and in addition $\boldsymbol{D}^*(x) = \boldsymbol{R}$, there exists a unique solution of the system*

$$\boldsymbol{y}_n(i) = \sum_{r=0}^{i-1} \boldsymbol{B}_n[r, \boldsymbol{y}_n(r)] + a, \quad \boldsymbol{y}_n(0) = a.$$

That solution exists is a consequence of Q, I and Q, II. $y_n(0)$ determines $y_n(1)$ which by Q, II is in region $\boldsymbol{R}$. $y_n(0)$, $y_n(1)$ determine $y_n(2)$, also in $\boldsymbol{R}$. It follows then by induction that $y_n(i)$ is in R and is unique.

THEOREM VII. *Given*

1. *The set of hypotheses Q,*
2. *$G_j(x, \boldsymbol{w}, \boldsymbol{w}') \leqq G_j(x)$, $G_j(x)$ a positive valued summable function on $(0, b)$,*
3. *$H_j[\boldsymbol{f}(x, \boldsymbol{a})]$ summable on $(0, b)$, $(j = 1, \cdots)$,*
4. *A method of fine subdivision of the interval $(0, b)$.*

Conclusion: $\lim_{n \to \infty} \boldsymbol{y}_n(x) = \boldsymbol{y}(x)$ uniformly in x on $(0, b)$, [$\boldsymbol{y}(x)$, $\boldsymbol{y}_n(x)$ solutions of the differential and difference systems respectively].

Proof. Consider a fixed value of x and for each value of n the subdivision containing x, i. e., $t_{in} \leqq x < t_{i+1n}$.

2.17 $\quad y_j(x) - y_{nj}(x) = \int_0^x \{f_j[t, \boldsymbol{y}(t)] - f_j[t, \boldsymbol{y}_n(t)]\} dt + \int_{t_{in}}^x f_j[t, \boldsymbol{y}_n(t)$

By hypothesis Q, 2.4, if $\boldsymbol{w}(x)$ and $\boldsymbol{w}'(x)$ are any pair of vectors on R,

$$\left| \int_{x_1}^{x_2} \{f_j[t, \boldsymbol{w}(t)] - f_j[t, \boldsymbol{w}'(t)]\} dt \right| \leqq M_j \int_{x_1}^{x_2} G_j(t)\, dt.$$

Taking $\boldsymbol{w}'(x) = \boldsymbol{a}$

$$\left| \int_{x_1}^{x_2} f_j[t, \boldsymbol{w}(t)] dt \right| \leqq \int_{x_1}^{x_2} [M_j G_j(t) + |f_j(t, \boldsymbol{a})|] dt.$$

Hence given any positive number $\epsilon > 0$, there exists an integer $n_{\epsilon j} > 0$ such

that for all values of $n > n_{\epsilon j}$, $\left| \int_x^{t_{in}} f_j[t, \mathbf{y}_n(t)]dt \right| < \epsilon$. Therefore

2.18 $$| y_j(x) - y_{nj}(x) | \leqq \int_0^x G_j(t) H_j[\mathbf{y}(t) - \mathbf{y}_n(t)]dt + \epsilon, \quad n > n_{\epsilon j}.$$

Also $$\mathbf{y}(x) - \mathbf{y}_n(x) = \int_0^x \{\mathbf{f}[t, \mathbf{y}(t)] - \mathbf{f}[t, \mathbf{y}_n(t)]\}dt + \int_{t_{in}}^x \mathbf{f}[t, \mathbf{y}_n(t)]dt.$$

2.19 $$H_j[\mathbf{y}(x) - \mathbf{y}_n(x)] \leqq \int_0^x K_j(t) H_j[\mathbf{y}(t) - \mathbf{y}_n(t)]dt + H_j\{\int_x^{t_{in}} \mathbf{f}[t, \mathbf{y}_n(t)]dt\}.$$

By hypothesis Q, line 2.5 if $\boldsymbol{w}(x)$ and $\boldsymbol{w}'(x)$ are any pair of vectors on $\boldsymbol{D}^*(x)$

$$H_j\{\int_{x_1}^{x_2} \mathbf{f}[t, \boldsymbol{w}(t)]dt - \int_{x_1}^{x_2} \mathbf{f}[t, \boldsymbol{w}'(t)]dt\} \leqq M_j \int_{x_1}^{x_2} K_j(t)dt.$$

Take $\boldsymbol{w}'(x) = \boldsymbol{a}$. Then by 1.8 and 1.9

$$H_j\{\int_{x_1}^{x_2} \mathbf{f}[t, \boldsymbol{w}(t)]dt\} \leqq M_j \int_{x_1}^{x_2} K_j(t)dt + H[\int_{x_1}^{x_2} \mathbf{f}(t, \boldsymbol{a})dt]$$

$$H_j\{\int_{x_1}^{x_2} [t, \boldsymbol{w}(t)]dt\} \leqq M_j \int_{x_1}^{x_2} K_j(t)dt + \int_{x_1}^{x_2} H_j[\mathbf{f}(t, \boldsymbol{a})]dt.$$

$\underset{x_1 \to x_2}{\text{Limit}}\ H_j\{\int_{x_1}^{x_2} [t, \boldsymbol{w}(t)]dt\} = 0$ uniformly in x and on the region R. Therefore given any positive number ϵ there exists a positive integer $n'_{\epsilon j}$ such that

2.20 $$H_j\{\int_{x_1}^{x_2} \mathbf{f}[t, \mathbf{y}_n(t)]dt\} < \epsilon \quad \text{when } n > n'_{\epsilon j}.$$

Take $n > n_{\epsilon j} + n'_{\epsilon j}$. Then

$$| y_j(x) - y_{nj}(x) | \leqq \int_0^x G_j(t) H_j[\mathbf{y}(t) - \mathbf{y}_n(t)]dt + \epsilon,$$

$$H_j[\mathbf{y}(x) - \mathbf{y}_n(x)] \leqq \int_0^x K_j(t) H_j[\mathbf{y}(t) - \mathbf{y}_n(t)]dt + \epsilon,$$

$$H_j[\mathbf{y}(x) - \mathbf{y}_n(x)] \leqq M_j \int_0^x K_j(t)dt + \epsilon,$$

$$H_j[\mathbf{y}(x) - \mathbf{y}_n(x)] \leqq M_{j/2!} [\int_0^x K_j(t)dt]^2 + [\int_0^x K_j(t)dt + 1]\epsilon.$$

By simple induction

2.21 $$H_j[\mathbf{y}(x) - \mathbf{y}_n(x)] \leqq \epsilon \exp [\int_0^b K_j(t)dt].$$

Then

2.22 $$| y_j(x) - y_{nj}(x) | \leqq \epsilon \{1 + \exp [\int_0^b K_j(t)dt] \int_0^b G_j(t)dt\}, \quad n > n_{\epsilon j} + n'_{\epsilon j}$$

and $\underset{n \to \infty}{\text{Limit}}\ y_{nj}(x) = y_j(x)$ uniformly in x on $(0, b)$. Therefore $\underset{n \to \infty}{\text{Limit}}\ \mathbf{y}_n(x) = \mathbf{y}(x)$.

Theorems V, VI, and VII are extensions of results obtained by H. J. Ettlinger * for linear difference and differential systems.

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* For abstract of paper see *Bulletin of the American Mathematical Society*, Vol. 34 (1928), p. 708.

Generalized Indecomposable Continua.

By P. M. SWINGLE.

1. *Introduction.* In this paper it is proposed to generalize some of the more useful definitions of indecomposable continua and to prove a number of theorems concerning the sets thus defined. The following definitions of indecomposable continua are known to be equivalent:

DEFINITION A. A continuum is *indecomposable* * when it is not the sum of two proper subcontinua (i. e. not the sum of two subcontinua different from itself).

DEFINITION B. A continuum is *indecomposable* † when each proper subcontinuum is a continuum of condensation.‡

DEFINITION C. A continuum is *indecomposable* § when it contains three points between any two of which it is irreducible.¶

In the generalization of the first two of these definitions the following definitions are useful:

DEFINITION. The set W is a *finished sum* of a set of subcontinua (M) of W if the sum of the subcontinua of (M) contain W but no M of (M) is contained in the sum of the remaining subcontinua of (M).

DEFINITION. A set $W_1, W_2, \cdots, W_k$ of subcontinua of a continuum W will be called a *convergable sequence of W under k* when $W_1 = W$, $W_{i-1} \neq W_i$ $(i = 2, \cdots, k)$ although W_{i-1} contains W_i, and $(W_{i-1} - W_i)' \neq W_{i-1}$ but for every subcontinuum C of W_k $(W_k - C)' = W_k$.

Definitions A and B can now be generalized as follows:

DEFINITION A. *A continuum W is indecomposable under index k,* where k is any positive integer, when W is the finished sum of k subcontinua of W but is not the finished sum of $k+1$ such subcontinua.

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* Z. Janiszewski and C. Kuratowski, "Sur les continus indécomposables," *Fundamenta Mathematicae*, Vol. 1, p. 210.

† Z. Janiszewski and C. Kuratowski, *loc. cit.*, theorem II, p. 212.

‡ If M is a point set, M' will be used to denote the set composed of M and the limit points of M. A subcontinuum K of a continuum C is said to be a *continuum of condensation of C* when $(C - K)' = C$.

§ Z. Janiszewski and C. Kuratowski, *loc. cit.*, theorem IV (III), p. 215.

¶ A continuum C is *irreducible between two points a and b* of C if no proper subcontinuum of C contains $a + b$.

DEFINITION B. A continuum W is *convergable under index k,* where k is any positive integer, when W contains a sequence convergable under k but does not contain a sequence convergable under a higher finite index.

It is seen that an indecomposable continuum is both indecomposable and convergable under index one. And it is readily seen that there exist examples which are both indecomposable and convergable under a higher index.

In this paper the impossibility of generalizing definition C in a certain manner will be shown. As it is based upon the idea of continua irreducible between two points an obvious possible generalization would be based upon the following definition:

DEFINITION. If N is a finite subset of a continuum M, then M will be said to be a *continuum irreducible between the points of N* when no proper subcontinuum of M contains N but there exist a proper subcontinuum of M containing any proper subset of N.*

While the proofs of the following theorems are given for the plane similar proofs can be given for a Euclidean space of n dimensions.

2. *Continua Indecomposable under Index k.*

LEMMA 1. *If Z is a connected subset of the connected set C such that $C-Z=C_1+C_2+\cdots+C_k$ separate,† then C_i+Z $(i=1,2,\cdots,k)$ is connected.*

Assume that $C_i+Z=X+Y$ separate. As Z is connected either X or Y contains it. Consider for example the case where X does. Then $C=(C_1+C_2+\cdots+C_{i-1}+C_{i+1}+\cdots+C_k+X)+Y$ separate which is a contradiction. Thus C_i+Z must be connected.

LEMMA 2. *If C is a circle, I is the interior of C, and M is a continuum then $C+M\times I$ is also a continuum; also every point of $M\times I$ is contained in a connected subset of $M\times I$ which has a limit point in C.*

It is evident that $C+M\times I=Z$ is closed. Assume that $Z=X+Y$ separate. Let $U=M-M\times I'$ and let $X\times M=x$ and $Y\times M=y$. As C is connected either X or Y contains it. Consider for example the case where X does. Then x contains $C\times M$ and $y\times C=0$. Thus $M=(U+x)+y$ separate which is a contradiction. Therefore $C+M\times I$ is a continuum. And since it is a bounded continuum every point of $M\times I$ must be contained in a connected subset of $M\times I$ which has a limit point in C.

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* See W. A. Wilson, "On the Separation of the Plane by Irreducible Continua," *Bulletin of the American Mathematical Society*, Vol. 33 (1927), p. 734.

† By $W=W_1+W_2+\cdots+W_k$ *separate* is meant that W is the sum of the distinct non-vacuous sets W_i no one of which contains a limit point of any other.

THEOREM 1. *If the continuum W, indecomposable under index k, is the finished sum of the set of subcontinua v_i $(i=1,2,\cdots,k)$ then $[v_i-(v_1+v_2+\cdots+v_{i-1}+v_{i+1}+\cdots+v_k)\times v_i]'$ is a subcontinuum of W.*

Let T_j $(j=1,2,\cdots,q;\ q<k)$ be the set of maximal connected subsets of $v_1+v_2+\cdots+v_{i-1}+v_{i+1}+\cdots+v_k=V_i$. Assume that $W-T_1=t_1+\cdots+t_p$ separate. Then it is necessary that p be less than $k+1$ for if not it is possible to take $p=k+1$ and then W is the finished sum of the $k+1$ subcontinua T_1+t_g $(g=1,2,\cdots,p)$, by lemma 1, which is contrary to the definition of a continuum indecomposable under index k. Hence $W-T_1$ can be taken as $t_1+t_2+\cdots+t_g$ separate, where every t_i is a connected set. As $W-T_1$ contains T_2 consider for example the case where t_1 contains T_2. Every point of $t_1'-t_1$ is a limit point of $v_i-V_i\times v_i$ as no point of T_1 is a limit point of any other T_j. Then just as above, $t_1'-T_2$ is the sum of a finite number, less than k, of maximal connected subsets which are closed except for limit points in T_2 and which each contain points of t_1 since these sets are separate. Proceeding in this manner it is seen that $(v_i-v_i\times V_i)'$ must be the sum of a finite number, less than k, of maximal connected subsets which are closed except for limit points in V_i. Thus $(v_i-V_i\times v_i)'$ must be a subcontinuum of W otherwise W is not indecomposable under index k.

COROLLARY 1. *If a continuum W is indecomposable under index k then W is the finished sum of a set of subcontinua u_i $(i=1,2,\cdots,k)$ of W where every point of u_i which is contained in $u_1+u_2+\cdots+u_{i-1}+u_{i+1}+\cdots+u_k=U_i$ is a limit point of $u_i-u_i\times U_i$.*

Since W is indecomposable under index k it is the sum of k subcontinua v_i. And by theorem 1 the points contained in only v_j of the v_i's plus the limit points of this point set is a continuum; that is $u_j=[v_j-v_j\times(v_1+v_2+\cdots+v_{j-1}+v_{j+1}+\cdots+v_k)]'$ is a subcontinuum of W. It is thus seen that W is the finished sum of the set of subcontinua u_i where every point of u_i which is contained in U_i is a limit point of $u_i-U_i\times u_i$.

THEOREM 2. *If W is indecomposable under index k then W is the finished sum of a set of k indecomposable subcontinua.*

As W is indecomposable under index k it follows by corollary 1 that W is the finished sum of a set of k subcontinua u_i $(i=1,\cdots,k)$ where $u_i=[W-(u_1+\cdots+u_{i-1}+u_{i+1}+\cdots+u_k)]'$. Thus if u_i is decomposable it follows that W is the finished sum of $k+1$ subcontinua which is a contradiction. Therefore u_i is indecomposable.

LEMMA 3. *If the continuum M, irreducible between two points a and b, is the finished sum of k indecomposable subcontinua u_i $(i = 1, \cdots, k)$ then* $u_i = [u_i - u_i(u_1 + \cdots + u_{i-1} + u_{i+1} + \cdots + u_k)]'$.

Since M is not the sum of $k-1$ of the u_i's, $(M - u_i)' \neq M$. Thus $(M - u_i)' = A + B$ * where A and B are either proper subcontinua of M or vacuous sets. Let C consist of the points of all those u_j's, except u_i, which have points common with A and let D similarly consist of those having points common with B. As $C + D$ does not contain u_i $C + D \neq M$. Hence $C \times D = 0$. But $(M - C)'$ is a continuum irreducible between two points † as is also $[(M - C)' - D]' = H$. But $H = [M - (C + D)]'$. Thus H is a subcontinuum of u_i but is not a proper subcontinuum of it for then $C + D$ contains u_i and so M is the sum of $k-1$ of the u_j's which is incorrect. Therefore $[M - (C + D)]' = [u_i - u_i \times (u_1 + \cdots u_{i-1} + u_{i+1} + \cdots + u_k)]'$ $= u_i$.

THEOREM 3. *If the continuum M, irreducible between two points a and b, is the finished sum of k indecomposable subcontinua u_i $(i = 1, \cdots, k)$ then M is indecomposable under index k.*

Assume that M is not indecomposable under index k. Then M is the finished sum of a set of subcontinua v_j $(j = 1, 2, \cdots, k+1)$. Let v be any one of the v_i's. The set v must contain points contained in only one of the u_i's for every point contained in more than one of the u_i's is a limit point of points contained in only one of the u_i's as shown in lemma 3. And if all of the points of v which are contained in only one u_i are contained in other v_i's then k of the v_i's contain M which is a contradiction under our assumption. Thus v contains a point q which is contained in only one of the u_i's, u say, and which is not contained in any other v_i. As M is irreducible between a and b $(M - u)'$ must be the sum of at most two maximal subcontinua.‡ Consider for example the case where it is the sum of two such subcontinua A and B. If $v \times (A + B) = 0$ then v is a proper subcontinuum of u and so $(u - v)' = u$ and so $(M - v)' = M$ and thus contains q which is a contradiction. Therefore either A or B contains a point of v. Consider for example the case where $A \times v \neq 0$. The set M contains an irreducible continuum T joining A and B.§ But u contains T and so $u = T$. Hence

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* C. Kuratowski, "Théorie des continus irréductibles entre deux points," ***Fundamenta Mathematicae***, Vol. 3, p. 202, theorem II.

† C. Kuratowski, *loc. cit.*, p. 204, theorem IV.

‡ C. Kuratowski, *loc. cit.*, theorem II, p. 202.

§ C. Kuratowski, *loc. cit.*, theorem VI, p. 205.

if $v \times B \neq 0$, v contains $T = u$ and so there would have to be as many u_i's as v_i's contrary to our assumption. Consider then the case where $v \times B = 0$. If then $(v - v \times A)'$ is a continuum it is a proper subcontinuum of u and so q is contained in some other v_i besides v which is a contradiction. Hence $(v - v \times A)' = X = X_1 + X_2$ separate. And as $A + v$ is a continuum $[M - (A + v)]' = Y$ is a continuum.* And $Y + A + X = M = Y + A + X_1 + X_2$. Also $A + X_j$ $(j = 1, 2)$ is a continuum as $(A + v - A)' = (v - v \times A)'$. Therefore either $A + X_1$ or $A + X_2$ contains a point of Y. Consider for example the case where $A + X_1$ does. The set Y does not contain X_j for if so $Y \times A \neq 0$ although $Y \times q = 0$. Hence $A + X_1 + Y$ is a proper subcontinuum of M containing $a + b$ which is impossible. Therefore M must be indecomposable under index k.

COROLLARY 2. *If M is irreducible between two points then a necessary and sufficient condition that M be indecomposable under index k is that M be the finished sum of a set of k indecomposable subcontinua of M.*

That the condition is necessary follows from theorem 2. And that it is sufficient follows from theorem 3.

THEOREM 4. *If the continuum M, irreducible between two points a and b, is indecomposable under index k, and N is a subset of M irreducibly connected between a and b, then N and $M - N$ are both dense in M.*†

As N' is a continuum irreducible between a and b, $N' = M$ and so N is dense in M.

Assume that $M - N$ is not dense in M. Since M is indecomposable under index k by theorem 2 M is the finished sum of a set of k indecomposable subcontinua u_i $(i = 1, \cdots, k)$. And by lemma 3 every point of M is a limit point of points contained in one of these u_i's alone. Hence since $(M - N)' \neq M$, $(M - N)'$ does not contain every point of M which is contained in only one of the u_i's. Let q then be a point contained only in $u_j = u$ and which is not contained in $(M - N)'$. Let R be a region containing q but such that R' does not contain a point of any u_i besides u nor does it contain a point of $(M - N)'$. Then by lemma 2 $M \times R$ contains a maximal connected subset T which contains q. Then both N and u contain T'. Thus as T' is a proper subcontinuum of u $(u - T')' = u$.

Since N is irreducibly connected between a and b, $N - T = N_1 + N_2$

* C. Kuratowski, *loc. cit.*, theorem III, p. 203.

† See P. Urysohn, "Mémoire sur les multiplicités Cantoriennes," *Fundamenta Mathematicae*, Vol. 8, p. 226, theorem 2.

$+\cdots+N_t$ separate where t is not less than two. Consider for example the case where N_1 contains a and N_2 contains b. Then by lemma 1 $N_1+T+N_2=N$ and hence $t=2$. Thus N_1 and N_2 are each connected.

As every point of T is a limit point of $u-T'$ and so of $N-T'$ every point of T is a limit point of either N_1 or N_2. Hence there must exist a point x which is a limit point of both. Thus $N_1+x+N_2=N$ which is a contradiction. Therefore $M-N$ is dense in M.

3. *Continua Convergable under Index k.* It is evident that if $W_1, W_2, \cdots, W_k$ is a convergable sequence of W under k where W is a continuum convergable under index k that W_j is convergable under index $k-(j-1)$.

THEOREM 5. *If W is convergable under index two then W is irreducible between two points and is the finished sum of two unbounded indecomposable subcontinua.*

Since W is convergable under index two there exists a convergable sequence $W_1=W, W_2$, of W under $k=2$. Hence $(W-W_2)'\neq W$ but every subcontinuum C of W_2 is such that $(W_2-C)'=W_2$. Therefore W_2 is indecomposable.

Let K be a proper subcontinuum of W such that W_2 is a proper subcontinuum of K if such a subcontinuum as K exists. Then $(W-W_2)'$ contains $(W-K)'$ and so $(W-K)'\neq W$ since $(W-W_2)'\neq W$. Thus every subcontinuum k of K is such that $(K-k)'=K$ and so $(K-W_2)'=K$. But $(W-W_2)'$ contains $(K-W_2)'$ and so $(W-W_2)'=W$ which is a contradiction. Therefore there does not exist a subcontinuum of W such as K.

Assume now that $W-W_2=X+Y$ separate. Then W_2+X is a proper subcontinuum such as K. Hence $W-W_2$ must be connected and so $(W-W_2)'$ is a proper subcontinuum of W. Assume that $(W-W_2)'=X+Y$ where X and Y are each proper subcontinua of $(W-W_2)'$. Then neither X nor Y contains $W-W_2$. Either X or Y must contain points of W_2. Consider for example the case where X does. Therefore $X+W_2$ is a proper subcontinuum such as K. As this is a contradiction $(W-W_2)'$ must be indecomposable. If $(W-W_2)'$ is bounded then it contains a strongly connected proper subset * T containing a point of W_2 such that W_2+T contains a subcontinuum such as K. Hence $(W-W_2)'$ must be an unbounded indecomposable proper subcontinuum of W.

* Z. Janiszewski and C. Kuratowski, "Sur les continus indécomposables," *Fundamenta Mathematicae*, Vol. 1, pp. 215-221.

As W_2 contains $[W-(W-W_2)']'$, $[W-(W-W_2)']' \neq W$. Therefore by the above reasoning $[W-(W-W_2)']'$ must also be an unbounded indecomposable proper subcontinuum of W. Hence W is the finished sum of the two unbounded indecomposable proper subcontinua $(W-W_2)'$ and $[W-(W-W_2)']'$.

Let a be any point of $(W-W_2)'$ which is not contained in $[W-(W-W_2)']'$ and let b be any point of $[W-(W-W_2)']'$ which is not contained in $(W-W_2)'$. Then W must be irreducible between a and b otherwise W contains a proper subcontinuum such as K.

As W_2 contains $[W-(W-W_2)']'$, $W_2=[W-(W-W_2)']'$ otherwise W_2 is a subcontinuum such as K. The following corollary then has been proven:

COROLLARY 3. *If W is convergable under index two and W, W_2 is a convergable sequence of W under two then W is the finished sum of the two unbounded indecomposable proper subcontinua $(W-W_2)'$ and $W_2=[W-(W-W_2)']'$.*

THEOREM 6. *If W is convergable under index k, $k \neq 1$, then W is the finished sum of k unbounded indecomposable subcontinua.*

Since W is convergable under index k W contains a sequence convergable under k, $W_1=W, W_2, \cdots, W_k$. Let $W_i=M_{k-(i-1)}$ $(i=1,2,\cdots,k)$. Then $W=M_k=(M_k-M_{k-1})'+(M_{k-1}-M_{k-2})'+\cdots+(M_3-M_2)'+(M_2-M_1)'+M_1$. Since M_2 is convergable under index two it is known by theorem 5 that $(M_2-M_1)'$ and M_1 are each unbounded indecomposable proper subcontinua of W. Let j be such that $(M_t-M_{t-1})'$ $(t=2,3,\cdots,j)$ is an unbounded indecomposable proper subcontinuum of W but this is not so if $t=j+1$. Assume that M_{j+1} contains a proper subcontinuum M which has M_j as a proper subcontinuum. Hence $(M_{j+1}-M_j)'$ contains $(M_{j+1}-M)'$ and $(M_{j+1}-M_j)'$ also contains $(M-M_j)'$. Thus as $(M_{j+1}-M_j)' \neq M_{j+1}$ $(M_{j+1}-M)' \neq M_{j+1}$ and $(M-M_j)' \neq M$. Thus M_{j+1} which is convergable under index $j+1$ contains the sequence, M_{j+1}, M, M_j, $M_{j-1}, \cdots, M_2$, M_1, convergable under $j+2$. As this is a contradiction a subcontinuum such as M does not exist. It is then necessary, just as shown in the proof of theorem 5, that $(M_{j+1}-M_j)'$ be a continuum but not either a decomposable or a bounded indecomposable continuum. Thus it must be an unbounded indecomposable proper subcontinuum of W. Hence $j=k$ and so W is the finished sum of a set of k unbounded indecomposable proper subcontinua.

THEOREM 7. *Let W be convergable under index k and let it also be the*

finished sum of a set of subcontinua M_i $(i=1,\cdots,k)$. *Let* K *be a subcontinuum of* W *which contains a point contained in* M_b *alone and also a point not contained in* M_b *alone. Then the points of* K *which are contained in* M_b *alone are contained in a finite number of subcontinua of* $K \times M_b$.

Let Z be composed of all the points of W which are not contained in M_b alone. Let Q be the maximal subcontinuum of $Z+K$ which contains K. As Z contains all M_i's except M_b, $Z \times Q$ contains at most $k-1$ maximal subcontinua. Let $C_1,\cdots,C_q$ be these maximal subcontinua. Assume that $Q-C_1=Q_1+Q_2+\cdots+Q_n$ separate where n is taken greater than k if possible. Let $W_1=W$, $W_2=Q=C_1+Q_1+\cdots+Q_n$, and $W_{n-t+2}=C_1+Q_1+\cdots+Q_t$ $(t=2,\cdots,n)$. If $Q \neq W$ then $(W-Q)' \neq W$ for W is a finished sum of the M_i's. And if $Q=W$, $(W-W_3)' \neq W$ for it equals Q_n'. Also $(W_i-W_{i+1})' \neq W_i$ since $(W_i-W_{i+1})'=Q_{n-i+2}'$. Hence since W is convergable under index k it is seen that $n=k$ at most. Thus $Q-C_1$ must be the sum of a finite number of maximal connected subsets. Let T be one of these maximal connected subsets such that T contains a C_i, C_2 say. Treating $T'-C_2$ in a similar manner it is seen that it must be the sum of a finite number of maximal connected subsets. Thus proceeding in this manner it is seen that $K-Z$ must be the sum of a finite number of maximal connected subsets of M_b from which fact the truth of the theorem follows.

THEOREM 8. *If* W *is convergable under index* k *then* W *is indecomposable under index* k.

By theorem 6 W is the finished sum of k indecomposable subcontinua M_i $(i=1,\cdots,k)$. Assume that W is also the finished sum of the set of subcontinua Z_j $(j=1,\cdots,p)$. Then Z_j contains a point q which is contained in it alone. But q must be contained in at least one M_i, M_n say. And from the proof of theorem 6 it is seen that every point of W is a limit point of the set of points which are contained in one M_i only. Thus Z_j must contain points which are contained in an M_n and Z_j only. Let x be such a point. Then by theorem 7, x is contained in a subcontinuum of $M_n \times Z_j$. Hence as every proper subcontinuum of an indecomposable continuum is a continuum of condensation it is necessary that Z_j contain M_n. Therefore p cannot be greater than k and so W is indecomposable under index k.

THEOREM 9. *Let* W *be convergable under index* k *and let it be the finished sum of the* k *unbounded indecomposable subcontinua* M_i. *Then no* M_j $(j=1,2,\cdots,i-1,\ i+1,\cdots,k)$ *contains a point of* k *maximal*

strongly connected proper subsets of M_i which contain also points of M_i alone.*

If the theorem is not true then M_j contains a point of k maximal strongly connected proper subsets of M_i and so there exist a subcontinuum N of W such that $(W - N)' \neq W$ although N contains a sequence convergable under k and so W contains a sequence convergable under index $k+1$. As this is a contradiction the theorem must be true.

THEOREM 10. *If W is convergable under index k then there exist a set of k points about which W is irreducible.*†

Let M_i be a set of k unbounded indecomposable subcontinua of which W is the finished sum. Let a_i be a point of M_i which is not contained in M_j $(j = 1, \cdots, i-1, i+1, \cdots, k)$ nor contained in a maximal strongly connected proper subset of M_i which has a point common with any M_j and there exist such points by theorem 9 and because of the fact that M_i contains an uncountable number of maximal strongly connected proper subsets. If W is not irreducible about the set $a_1 + \cdots a_k$ then there exist a proper subcontinuum K of W which contains it. Therefore by theorem 7, since K cannot contain all of some M_i, M_c say, a_c is contained in a maximal strongly connected proper subset of M_c which contains a point of some M_j, $j \neq c$, which is a contradiction. Hence W is irreducible about k points.

COROLLARY 4. *If W is convergable under index k then there exist q points, q not greater than k, between which W is irreducible.*

There exists by theorem 10 a set of k points about which W is irreducible. It is then evident that this set of points must contain a subset between the points of which W is irreducible.

4. *Continua Irreducible Between k Points.* As stated in definition C it has been shown that in order that a continuum M be indecomposable it is necessary and sufficient that M contain a set N of $k+1$ points, where $k = 2$, such that M is irreducible between any combination of k of these points. Here it will be shown that there does not exist a set having this property for k greater than two.

By definition C it is seen that an indecomposable continuum is irreducible between two points. A number of theorems have been proven giving neces-

* See Z. Janiszewski and C. Kuratowski, "Sur les continus indécomposables," *Fundamenta Mathematicae*, Vol. 1, p. 218.

† The continuum W is said to be irreducible about the set N of W if no proper subcontinuum of W contains N.

sary and sufficient conditions that an irreducible continuum be indecomposable. Here the impossibility of generalizing these, in a certain manner, is shown.

In the remainder of this paper let $N = a_1 + a_2 + \cdots + a_k$ and let M be a continuum irreducible between the points of N. If M is also irreducible between the points of $N - a_i + a$ then a will be called an *a_i co-end-point of M*. The set composed of all the a_i co-end-points of M will be called the *a_i co-end-set of M* and will be represented by $(a_i)_{N-a_i}$ or by (a_i).*

It has been shown that if $k = 2$ a necessary and sufficient condition that the irreducible continuum M be indecomposable is that $(a_1) \times (a_2) \neq 0$.† However this is not true for k greater than two as shown by the following theorem.

THEOREM 11. *If k is greater than two then $(a_i) \times (a_j) = 0$.*

Assume that $(a_i) \times (a_j)$ contains a point a. Let X be any subcontinuum of M containing $N - a_i$ and let Y be any subcontinuum containing $N - a_j$ such that $X \neq M \neq Y$. But $X + Y$ contains N and so $X + Y = M$. Hence either X or Y contains a in which case either $X = M$ or $Y = M$ which is a contradiction. Thus $(a_i) \times (a_j) = 0$.

THEOREM 12. *If Y is a proper subcontinuum of M containing $N - a_i$ then $(M - Y)'$ is a continuum irreducible between a_i and any point of $Y \times (M - Y)'$.*

Assume that $M - Y = X + Z$ separate. Either X or Z contains a_i. Say that Z does. Then the continuum $Z + Y$ contains N and so $Z + Y = M$ which is a contradiction. Hence $M - Y$ is connected and so $(M - Y)'$ is a subcontinuum of M. It is then evident that $(M - Y)'$ is irreducible between a_i and any point of $Y \times (M - Y)'$ otherwise M is not irreducible between the points of N.

THEOREM 13. *If M is irreducible about the points of the finite set Z then $Z \times (a_i) \neq 0$.*

Assume that $Z \times (a_i) = 0$. Let Y_i be composed of all those points of M which are contained in proper subcontinua of M containing $N - a_i$. Let z be any point of Z. There exists a proper subcontinuum of M containing $N - a_i + z$ otherwise $(a_i) \times Z$ contains z. Hence Y_i contains Z. And since Z contains a finite number of points, n say, there exist at least n sub-

* See C. Kuratowski, "Théorie des continus irréductibles entre deux points II," *Fundamenta Mathematicae*, Vol. 10, p. 230. See also P. M. Swingle, "End-sets of continua irreducible between two points," *Fundamenta Mathematicae*.

† This follows readily from the work of S. Mazurkiewicz, Z. Janiszewski and C. Kuratowski. See however P. M. Swingle, *loc. cit.*

continua of Y_i whose sum contains $Z + N - a_i$ and whose sum then equals M. Thus $Y_i = M$ which is a contradiction, as $Y_i \times a_i = 0$. Therefore $Z \times (a_i) \neq 0$.

THEOREM 14. *If M is irreducible about the finite point set Z then Z contains a subset of k points between the points of which M is irreducible.*

Since M is irreducible about the finite point set Z, the set Z must contain a subset X between the points of which M must be irreducible. Let x be the number of points in X. By theorem 13 $X \times (a_i) \neq 0$. And by theorem 11 $(a_i) \times (a_j) = 0$. Hence X contains at least k points and similarly N must contain at least x points. Hence $x = k$.

COROLLARY 5. *If Y is a proper subcontinuum of M containing $N - a_i$ then, if k is greater than two, $(M - Y)' \neq M$.*

For if $(M - Y)' = M$ then $(M - Y)'$ is irreducible between k points and by theorem 12 it is also irreducible between two points. Hence by theorem 14 $k = 2$.

COROLLARY 6. *If k is greater than two then $(a_i)' \neq M$.**

Let Y be a proper subcontinuum of M containing $N - a_i$. Then $Y \times (a_i) = 0$. Hence $(M - Y)$ contains (a_i) and so $(M - Y)'$ contains $(a_i)'$. Thus by corollary 5 $(a_i)' \neq M$.

THEOREM 15. *If k is greater than two and Y is a proper subcontinuum of M containing $N - a_i$ then Y contains (a_j) $(j \neq i)$.*

Assume that Y does not contain a of (a_j). Then $(M - Y)'$ contains a. Let K be a proper subcontinuum of M containing $N - a_j$. As k is greater than two $(N - a_i) \times (N - a_j) \neq 0$ and so $K \times Y \neq 0$. Hence $K + Y$ is a subcontinuum of M containing N and so $K + Y = M$. Thus K contains $(M - Y)'$ and so contains a. Hence K contains $a + N - a_j$ and so $K = M$ which is a contradiction. Therefore Y contains a and so contains (a_j).

It has been shown that, for $k = 2$, a necessary and sufficient condition that the irreducible continuum M be indecomposable is that $(a_1) + (a_2) = M$.† The following theorem shows the impossibility of generalizing this for the cases where k is greater than two.

THEOREM 16. *If k is greater than two $(a_1) + (a_2) + \cdots + (a_k) \neq M$.*

* For $k = 2$ C. Kuratowski has proven that $(a_i)' = M$ is a necessary and sufficient condition that the irreducible continuum M be indecomposable. See C. Kuratowski, *loc. cit.*, *Fundamenta Mathematicae*, Vol. 10, p. 235, theorem I.

† P. M. Swingle, *loc. cit.*

Assume that $(a_1)+\cdots+(a_k)=M$. Let Y be a proper subcontinuum of M containing $N-a_i$. Then $Y\times(a_i)=0$. By theorem 15 Y contains the sum of the (a_j)'s $(j\neq i)$. Hence $Y=(a_1)+\cdots+(a_{i-1})+(a_{i+1})+\cdots+(a_k)$. By theorem 11 $(a_i)\times(a_j)=0$. Hence $M-Y=(a_i)$. Thus (a_i) $(i=1,\cdots,k)$ is not closed while the sum of $k-1$ of the (a_j)'s is closed. As this is a contradiction the theorem is true.

It is possible however to generalize the (a_i) co-end-set for $k=2$ in another manner. Let $Z_i=N-a_i$ and let $(Z_i)_{a_i}=(Z_i)$ be the set of point sets which have the property that together with a_i M is irreducible between these points. The set (Z_i) will be called the *Z_i co-end-set of M.*

It is evident that for $k\neq 2$ $(Z_i)\times(Z_j)\neq 0$ *for every M.*

THEOREM 17. *If Y is a proper subcontinuum of M containing the set Z of (Z_i) then Y contains (Z_i), if k is greater than two.*

Let z be a set of (Z_i) and b a point of Z. Then by theorem 13 $z\times(b)\neq 0$. And by theorem 15 Y contains (b). Hence as the (b)'s contain z, Y contains (Z_i).

THEOREM 18. *If k is greater than two, $(Z_1)+(Z_2)+\cdots+(Z_k)\neq M$.*

Let Z be a set of (Z_i). Then M is irreducible between the points of $Z+a_i$. Then $(a_1)+\cdots+(a_k)$ contains $Z+a_i$ by theorems 13 and 14. Hence $(a_1)+\cdots+(a_k)\neq M$ contains $(Z_1)+\cdots+(Z_k)$ and so $(Z_1)+\cdots+(Z_k)\neq M$.

THEOREM 19. *If B is a set of $k+1$ points of a continuum C, where $k\neq 1$ or 2, then C is not irreducible between every combination of k of the points of B.*

Assume that C is irreducible between every combination of k of the points of B. Then C is not irreducible between the combination of any $k-1$ of the points of B for assume that Z is a set of $k-1$ of these points. Then by theorem 14 Z must contain k points. Thus each combination of $k-1$ of the points of B is contained in a proper subcontinuum K of M. Consider the sum S of a set of such K's where S contains and only contains a K for each combination of the points of $B-b_{k+1}$, where $B=b_1+\cdots+b_{k+1}$, taken $k-1$ at a time and where b_1 is contained in each of these combinations. Then S is a subcontinuum of C containing k points of B. Hence $S=M$. Thus one of the K's, K_1 say, contains b_{k+1} and so contains k of the points of B. Therefore $K_1=M$ which is a contradiction. Hence C is not irreducible between every combination of the points of B taken k at a time.

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Differential Equations Containing Absolute Values of Derivatives.

By Cletus Odia Oakley.

1. *Introduction.* In this paper we consider the non-linear differential equation

$$u'' + p_1 u' + p_2 u + q_1 \mid u' \mid + q_2 \mid u \mid = \phi, \tag{1}$$

where p_1, p_2, q_1, q_2, ϕ are real, single-valued and continuous functions of the real variable x throughout an interval (a, b); and investigate the questions of existence and nature of real solutions, separation of zeros, comparison and oscillation theorems.* Although equation (1) is non-linear, it will be referred to as a linear differential equation with absolute values for, as we shall see, the solutions of it follow those of a set of related linear equations, namely:

$$\begin{aligned}
&(2_1)\ u'' + p_1 u' + p_2 u + q_1 u' + q_2 u = \phi, \quad u(+),\dagger\ u'(+);\\
&(2_2)\ u'' + p_1 u' + p_2 u - q_1 u' + q_2 u = \phi, \quad u(+),\ \ u'(-);\\
&(2_3)\ u'' + p_1 u' + p_2 u - q_1 u' - q_2 u = \phi, \quad u(-),\ \ u'(-);\\
&(2_4)\ u'' + p_1 u' + p_2 u + q_1 u' - q_2 u = \phi, \quad u(-),\ \ u'(+).
\end{aligned} \tag{2}$$

These equations will be known as the associated equations or the set associated with (1).

In § 2 we establish the existence of solutions of (1) with continuous first and second derivatives. We also lay down certain definitions, call attention to the first order homogeneous equation and state without derivation two theorems which will be of use in the development of the subject and which are readily established. The case of constant coefficients is treated in § 3, where, in view of the richness of the material, considerable space is devoted to the theory centering around the homogeneous equation. In § 4 we treat the case of variable coefficients where the equation is again homogeneous, deriving identities and theorems which are in some respects similar to those of the Sturmian theory associated with linear equations. While the treatment is such that the classic theory can be obtained by making identically zero the coefficients of the terms involving absolute values, yet it should be pointed out that certain relations (which exist when these coefficients are

* See Bôcher, "Leçons sur les Méthodes de Sturm," where further references to the linear theory will be found.

† By a symbol $\mu(+)$ we shall mean μ is positive-valued.

identically zero) can not exist when the absolute-value terms are actually present and hence that much of the theory herein treated has no counterpart in the classic material. It is in this section that the prnicipal results of the paper occur.

2. *Existence and Nature of Solutions.* We begin with the following

DEFINITION. *A function $u(x)$, continuous in its first two derivatives in the interval (a, b), which satisfies the equation* (1) *will be said to be a solution of* (1) *in that interval.*

The interval may be open or closed. A solution of the first order equation would be defined in like manner. It will be instructive to begin the discussion of the existence and nature of solutions by examining the first order equation

$$u' + pu + q \mid u \mid = \phi \tag{3}$$

and its two associated equations

$$\begin{aligned} &(4_1) \quad u' + pu + qu = \phi, \\ &(4_2) \quad u' + pu - qu = \phi. \end{aligned} \tag{4}$$

In the first place, if $\phi \equiv 0$, a solution u_1 of (4_1) such that $u_1(\gamma) = \delta(+)$, where γ is any point in the interval (a, b), will remain positive throughout (a, b) and will, moreover, satisfy (3). In a similar manner a solution u_2 of (4_2) with initial conditions $u_2(\gamma) = \delta(-)$ will remain negative and will be a solution of (3). Now suppose $\phi \not\equiv 0$ and consider u_1 of (4_1) such that $u_1(x_1) = \delta(+)$. To the right of x_1 and so long as it remains positive, this function $u_1(x)$ will satisfy (3). Let $u_1(x_2) = 0$, $\phi(x_2) \neq 0$, x_2 being the first vanishing point of u_1 to the right of x_1. A solution $u_2(x)$ of (4_2)* with the condition $u_2(x_2) = 0$ will match up with u_1 with continuous derivative for at x_2 both (4_1) and (4_2) reduce to $u'(x_2) = \phi$, with $u(x_2) = 0$. Continuing the process of piecing together these two functions $u_1(x)$ and $u_2(x)$, we obtain a solution $u(x)$, $u(\gamma) = \delta$, of (3), throughout (a, b).

We turn now to equation (1) and the set (2) and confine our attention to what will be a typical example of the manner in which a solution of (1) may be obtained by matching up the solutions of (2). Consider, then, a solution u_1 of (2_1) with initial conditions $u_1(x_1) = 0$, $u_1'(x_1) = \delta'(+)$. So long as u_1' remains positive, u_1 will also be a solution of (1). Equation (1), therefore, possesses a solution at least in the open interval (x_1, x_2) where

* The discussion here is typical rather than complete. If $\phi(x_2) = 0$ so that $u_1'(x_2) = 0$, it may happen that $u_1(x)$ of (4_1) continues to be a solution of (3).

x_2 is the first vanishing point of u_1' to the right of x_1. Beyond this point the solution of (2_1) will continue to be a solution of (1) if and only if u_1' remains positive in the neighborhood of x_2. We suppose, then that u_1' is negative to the right of x_2 and consider a solution u_2 of the equation (2_2) such that $u_2(x_2) = u_1(x_2)$, $u_2'(x_2) = 0$. It is clear that u_2 will be a solution of (1) if and only if u_2' is negative in the right neighborhood of x_2. In order to show that u_2' is negative to the right of x_2, we notice first that at x_2 we have, from equation (2_1), $u''(x_2) + p_2u(x_2) + q_2u(x_2) = \phi$ and that $u_1''(x_2)$ is negative. Also, at this point, we get, from equation (2_2), $u''(x_2) + p_2u(x_2) + q_2u(x_2) = \phi$ and the solution u_2 defined above is such that $u_2(x_2) = u_1(x_2)$, $u_2'(x_2) = u_1'(x_2) = 0$. Hence $u_2''(x_2)$ is also negative and the solution $u(x)$ with continuous first and second derivatives of equation (1) persists, through the medium of (2_1) and (2_2), in the open interval (x_1, x_3) where x_3 is the point to the right where first u_2 or u_2' becomes zero. If u_2' is the first to become zero, then the solution $u(x)$ of (1) may switch back to that of (2_1); but if u_2 is the first to become zero, then, if we consider a solution u_3 of (2_3) such that $u_3(x_3) = 0$, $u_3'(x_3) = u_2'(x_3)$, a repetition of the argument carries the solution of (1) on by means of equation (2_3) and further, perhaps,* by equation (2_4). Hence, by means of the solutions of the associated set (2), the existence of a unique solution with continuous first and second derivatives of equation (1)† is demonstrated. Moreover this is in essence a general solution according to the following

DEFINITION. *A solution $u(x)$ of (1) will be called a general solution in (a, b) if $u(x)$ contains arbitrary elements such that for every x_0 in (a, b) and for any whatever given set of constants δ, δ' it is possible to specialize those arbitrary elements so as to make $u(x_0) = \delta$, $u'(x_0) = \delta'$.*

For suppose we take any point γ, fixed in (a, b), and contemplate the set of solutions $v(x)$, $v(\gamma) = \sigma$, $v'(\gamma) = \sigma'$, where σ, σ' are arbitrary real numbers; and further consider any point x_0 in (a, b) and two constants δ, δ'. Now we have just shown that there exists a solution $u(x)$, $u(x_0) = \delta$, $u'(x_0) = \delta'$, which, if followed to the point γ, will be such that $u(\gamma) = \sigma_1$,

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* A solution of (1) could, of course, follow those of (2_1), (2_2) and (2_3) only; or of (2_1) and (2_2) only; etc.

† By similar methods the existence of a solution continuous in its first n-derivatives of the equation

$$(1) \qquad u^{(n)} + p_1u^{(n-1)} + \cdots + p_nu + q_1 \mid u^{(n-1)} \mid + q_2 \mid u^{(n-2)} \mid + \cdots + q_n \mid u \mid = \phi$$

could be demonstrated.

$u'(\gamma)=\sigma_1'$. Hence by specializing σ, σ' to σ_1, σ_1', we obtain from $v(x)$ the solution $u(x)$, $u(x_0)=\delta$, $u'(x_0)=\delta'$. Thus $v(x)=f(\gamma;\ \sigma,\ \sigma',\ x)$ is the general solution and the arbitrary elements are σ, σ'. This does not, however, give us the desired information concerning the way in which the arbitrary elements enter into the general solution. We notice that even if u_1 is a solution of the homogeneous equation

$$u'' + p_1u' + p_2u + q_1 \mid u' \mid + q_2 \mid u \mid = 0, \tag{1'}$$

it does not follow that ku_1, where k is a constant,* will be a solution; nor will the sum of two solutions $(u_1 + u_2)$ in general be a solution. Although the arbitrary elements evidently do not enter linearly, the notion of linearly dependent solutions is an important one. We have immediately the

THEOREM I. *A necessary and sufficient† condition that two solutions of* (1') *be linearly dependent is that the wronskian vanish identically.*

By methods that are standard in the treatment of linear equations the following theorem can be established.

THEOREM II. *Any solution of* (1') *which together with its first derivative vanishes at a point* γ, *vanishes identically.*‡

We shall exclude this solution from consideration throughout this paper.

The general, positive-valued solution u_1 of the homogeneous equation

$$u' + pu + q \mid u \mid = 0 \tag{3'}$$

is

$$u_1 = c_1 \exp\left[-\int_{c_0}^{x} \{p(\xi) + q(\xi)\}\, d\xi\right]$$

where c_1 is a positive constant and c_0 is any point in (a, b); and the general, negative-valued solution u_2 is

$$u_2 = c_2 \exp\left[-\int_{c_0}^{x} \{p(\xi) - q(\xi)\}\, d\xi\right]$$

where c_2 is a negative constant. Hence

$$u_2 = (c_2/c_1) \exp\left[2\int_{c_0}^{x} q(\xi)\, d\xi\right] u_1.$$

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* In general ku_1 will be a solution of (1') when and only when k is positive.

† See Bôcher, "Certain Cases in Which the Vanishing of the Wronskian is a Sufficient Condition for Linear Dependence," *Transactions of the American Mathematical Society*, Vol. 2 (1901), p. 139. Theorem I (p. 140) shows that $u_1 = cu_2$ provided $u_2 \neq 0$. If $u_2 = 0$, then $u_1 = 0$ necessarily and we may use the same constant c. Finally, from the continuity of u_1', u_2' (in particular at the zeros of u_2), it follows that we must have the same c throughout (a, b) even though u_2 changes sign in (a, b).

‡ Theorems corresponding to I and II could be formulated for equation (1), $\phi \equiv 0$, in footnote †, page 661.

The general, complex-valued solution $z(x) = u(x) + iv(x)$, $u(x)$ and $v(x)$ real, and $v(x) \not\equiv 0$, is *

$$z = u + iv = k_1 \exp\left[-\int_{c_0}^{x} \{p(\xi) + q(\xi)\}d\xi\right]$$
$$-(k_2^2/k_1)\exp\left[-\int_{c_0}^{x} \{p(\xi) - q(\xi)\}d\xi\right] + 2ik_2 \exp\left[-\int_{c_0}^{x} p(\xi)d\xi\right]$$

where $k_1 > 0$ and k_2 is real and not zero. We call attention to the fact that the first term of z is the general, positive-valued solution, that the second term is the general, negative-valued solution, and that the last term is a solution of the ordinary linear equation. If $q \not\equiv 0$, equation (3′) can not have a pure imaginary solution.

3. *The Case of Constant Coefficients.* We write down the equation

$$u'' + a_1u' + a_2u + b_1 \mid u' \mid + b_2 \mid u \mid = 0 \tag{5}$$

and the associated set

$$\begin{aligned}
&(6_1) \quad u'' + a_1u' + a_2u + b_1u' + b_2u = 0\\
&(6_2) \quad u'' + a_1u' + a_2u - b_1u' + b_2u = 0\\
&(6_3) \quad u'' + a_1u' + a_2u - b_1u' - b_2u = 0\\
&(6_4) \quad u'' + a_1u' + a_2u + b_1u' - b_2u = 0.
\end{aligned} \tag{6}$$

Along with (5) and (6) we shall need to consider the quantities

$$\begin{aligned}
m_{11}, \dagger\ m_{12} &= 1/2\ \{-(a_1 + b_1) \pm [(a_1 + b_1)^2 - 4(a_2 + b_2)]^{1/2}\}\\
m_{21}, m_{22} &= 1/2\ \{-(a_1 - b_1) \pm [(a_1 - b_1)^2 - 4(a_2 + b_2)]^{1/2}\}\\
m_{31}, m_{32} &= 1/2\ \{-(a_1 - b_1) \pm [(a_1 - b_1)^2 - 4(a_2 - b_2)]^{1/2}\}\\
m_{41}, m_{42} &= 1/2\ \{-(a_1 + b_1) \pm [(a_1 + b_1)^2 - 4(a_2 - b_2)]^{1/2}\}\\
\Delta_1 &= (a_1 + b_1)^2 - 4(a_2 + b_2)\\
\Delta_2 &= (a_1 - b_1)^2 - 4(a_2 + b_2)\\
\Delta_3 &= (a_1 - b_1)^2 - 4(a_2 - b_2)\\
\Delta_4 &= (a_1 + b_1)^2 - 4(a_2 - b_2)
\end{aligned}$$

which are the roots and discriminants respectively of the characteristic equations corresponding to the set (6). If $\Delta_j < 0$, we write $m_{j1} = \alpha_j + i\beta_j$,

* If $z = u + iv$ is to satisfy (3′) we must have, (a) $v' + pv = 0$, (b) $u' + pu + q(u^2 + v^2)^{1/2} = 0$ simultaneously. Equation (a) is immediately integrable. Equation (b) represents a particular type of non-linear equation in which, because of the peculiar way in which v enters, the variables are made separable by making the transformation $u = gt$ and afterwards choosing g in such wise as to satisfy the equation $g' + pg = 0$.

† Read m_{j1} with + sign, m_{j2} with − sign; $(j = 1, 2, 3, 4)$.

$m_{j2} = \alpha_j - i\beta_j$, $j = 1, 2, 3, 4$; $i = (-1)^{\frac{1}{2}}$. We shall designate by y_j, v_j, s_j, t_j, real solutions of (6_j) as follows:

(Exp.) s_j $\begin{cases} (7) & y_j = c_1 e^{m_{j1}x} + c_2 e^{m_{j2}x}, \quad \Delta_j > 0, \ m_{j1} \neq m_{j2} \\ (8) & v_j = c_1 e^{m_j x} + c_2 x e^{m_j x}, \quad \Delta_j = 0, \ m_{j1} = m_{j2} = m_j \end{cases}$

(Trigonometric) (9) $t_j = e^{\alpha_j x} c_1 \sin(\beta_j + c_2)$, $\Delta_j < 0$,

where c_1, c_2 are the arbitrary elements and s_j represents either exponential function y_j or v_j.

It is convenient at this point to consider some properties of (7) and (8) that are of fundamental importance to our problem. In the first place, suppose in (7) that m_{j1}, and m_{j2} are of the same sign. Then y_j, with appropriate initial conditions to make c_1 and c_2 of one and the same sign, will represent a solution of (5) throughout any finite interval (a, b). This follows at once because in this case neither y_j nor y_j' is ever zero and this solution of (5) with these boundary conditions remains forever on a solution of one and the same associated equation (6_j). If boundary conditions necessary to make c_i, c_2 of opposite sign are imposed, then y_j vanishes at one and only one point x_0 and y_j' vanishes not at all. In this case y_j will represent a solution of (5) to the right (or left) of x_0. Similarly, if m_{j1} and m_{j2} are of opposite sign, then a solution y_j of (7), with initial conditions such that c_1 and c_2 are of opposite sign, will vanish at no point while its derivative y_j' vanishes at one and but one point x_1. For y_j to vanish at one and only one point x_0 and also for y_j' to vanish at one and only one point x_1 it is both necessary and sufficient that c_1, c_2 be of opposite sign and that at the same time m_{j1}, m_{j2} be of the same sign. Further, if m_{j1}, m_{j2} are negative, then x_0 will lie to the left of x_1; if m_{j1}, m_{j2} are positive, x_0 will lie to the right of x_1. Moreover, the distance $(x_1 - x_0) = (\log m_{j1} - \log m_{j2})/(m_{j2} - m_{j1})$, is not zero and is independent of c_1, c_2. Examining v_j and assuming $c_2 m_j \neq 0$, we note: that there always exist two and only two points x_0 and x_1 such that $v_j(x_0) = 0$, $v_j'(x_1) = 0$; that if m_j is negative, x_0 will lie to the left of x_1; that if m_j is positive, x_0 will lie to the right of x_1; that $(x_1 - x_0) = 1/m_j$, is not zero and is independent of c_1, c_2. Hence any function $s_j(x)$ of the form (7) or (8) will possess the property that $s_j(x_0) = 0$, $s_j'(x_1) = 0$, provided that m_{j1}, m_{j2} are of the same sign and c_1, c_2 are of opposite sign.

With these things in mind, we seek conditions on the coefficients of (5) so that a solution $u(x)$, $u(\gamma) = \delta(+)$, $u'(\gamma) = \delta'(+)$ of (5) shall be made up entirely of solutions $s_j(x)$ and which shall have as many zeros as possible in the interval from minus infinity to plus infinity. We shall consider that δ, δ' have been so chosen that c_1, c_2 are of opposite sign. Two things

are evident at once:—First: In order that zeros of u exist to the left of γ, s_1 must vanish to the left of γ, say $s_1(x_0)=0$, where $x_0<\gamma$. Second: In order that zeros of u exist to the right of γ, s_1' must vanish to the right of γ, say $s_1'(x_1)=0$, $x_1>\gamma$. And it follows immediately from the considerations above that m_{11} and m_{12} must both be of the same sign and negative. Assuming that m_{11} and m_{12} are negative, we follow this function s_1 up to the point x_1 and there connect with a solution of (6_2).

The solution $u(x)$ of (5), then, will follow s_1 up to x_1 where it will join to $s_2(x)$ with initial conditions $s_2(x_1)=s_1(x_1)$, $s_2'(x_1)=s_1'(x_1)=0$. If s_2 is to vanish to the right of x_1, say at x_2, it follows that m_{21}, m_{22} must both be positive. We assume this condition on m_{21}, m_{22} and note that this gives $u(x)$ two zeros so far—one at x_0 and one at x_2. In order for s_3 to join with s_2 at x_2 and to be such that $s_3'(x_3)=0$, $x_3>x_2$, it is necessary that m_{31}, m_{32} be negative. Now if s_4 matches up with s_3 at x_3 and vanishes to the right at x_4, say, it is obvious that s_1, s_2, s_3, s_4 will continue to match up throughout any finite interval (a, b) and form an oscillating solution $u(x)$ of (5). But the condition for s_4' to vanish at x_3 and for s_4 to vanish at x_4 is that m_{41}, m_{42} be positive and this is impossible in view of the conditions already imposed on m_{11}, m_{12}, m_{21}, m_{22}, m_{31}, m_{32}. We have therefore established the following

THEOREM III. *There exists no solution $u(x)$ of (5) made up wholly of exponential parts $s_j(x)$ which vanishes more than twice in any interval (a, b).*

Because of the fact that a solution of (5) may possess two and only two zeros we lay down the following

DEFINITION. *A solution $u(x)$ of (5) will be said to oscillate in an interval (a, b) if, in that interval, it vanishes as many as three times.*

While the totality of conditions

$$(10) \qquad m_{1\nu}(-),\ m_{2\nu}(+),\ m_{3\nu}(-),\ m_{4\nu}(+), \qquad \nu=1,2$$

is not realizable, yet the *order of the signs exhibited* is of highest importance. For suppose we follow a solution $u(x)$ as we did (in the above discussion) up to the point x_3 and ask if at this point the solution could be continued by t_4 of trigonometric form (9)? An examination of the m's shows that Δ_4 can not be less than zero when $m_{11}(-)$, $m_{12}(-)$, $m_{21}(+)$, $m_{22}(+)$, $m_{31}(-)$, $m_{32}(-)$. It is not difficult to show that it is impossible for any Δ_j to be negative when the other m's follow the law of signs (10). Hence the

THEOREM IV. *There exists no oscillating solution $u(x)$ of (5) made up of three exponential parts and one trigonometric part.*

Suppose, now, $m_{11}(-)$, $m_{12}(-)$, $m_{21}(+)$, $m_{22}(+)$ and that we have followed through on a solution u up to the point x_2. If $\Delta_3 < 0$, a solution t_3 of (6_3) and of form (9) will connect with s_2 at x_2. It is obvious that t_3' will vanish to the right of x_2 at some point which we shall call x_3. Further, if $\Delta_4 < 0$, t_4 will match up with t_3 at x_3 and will vanish to the right at x_4 from which point, it is apparent, s_1, s_2, t_3, t_4 will continue to match up to form an oscillating solution of (5). This establishes the

THEOREM V. *There exists a class of oscillating solutions $u(x)$ of* (5) *members of which are made up of two exponential parts (where u is positive-valued) and two trigonometric parts (where u is negative-valued).* It may easily be verified that: *There exists another class of oscillating solutions $u(x)$ of* (5) *members of which are made up of two exponential parts (where u is negative-valued) and two trigonometric parts (where u is positive-valued), and that these are the only two classes of oscillating solutions containing two exponential parts.*

By the same methods the following two theorems are obtained:

THEOREM VI. *There exist four classes of oscillating solutions $u(x)$ of* (5) *members of which are made up of three trigonometric parts and one exponential part.*

THEOREM VII. *There exists one class of oscillating solutions $u(x)$ of* (5) *members of which are made up wholly of trigonometric parts.*

We list the possibilities and conditions for the sake of reference:

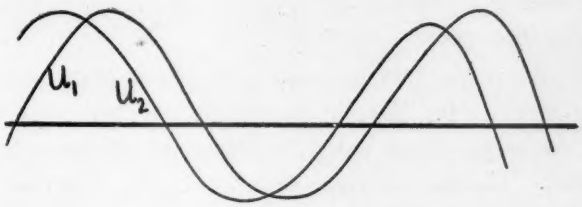
FOR OSCILLATING SOLUTIONS OF (5).

Type	Conditions	Number of Classes of Solutions
I. Four trigonometric parts.	$\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 < 0,\ \Delta_4 < 0.$	1.
II. Three trigonometric parts, One exponential part.	$\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 < 0,\ m_{4\nu}(+)$; $\Delta_1 < 0,\ \Delta_2 < 0,\ m_{3\nu}(-), \Delta_4 < 0$; $\Delta_1 < 0,\ m_{2\nu}(+), \Delta_3 < 0,\ \Delta_4 < 0$; $m_{1\nu}(-), \Delta_2 < 0,\ \Delta_3 < 0,\ \Delta_4 < 0.$	4.
III. Two trigonometric parts, Two exponential parts.	$\Delta_1 < 0,\ \Delta_2 < 0,\ m_{3\nu}(-), m_{4\nu}(+)$; $m_{1\nu}(-), m_{2\nu}(+), \Delta_3 < 0,\ \Delta_4 < 0.$	2.

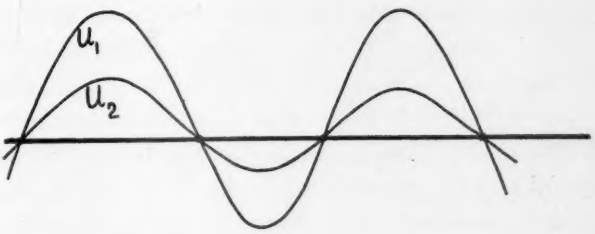
Now in general the distance between consecutive zeros between which a solution is positive-valued will not be the same as the distance between consecutive zeros between which the solution is negative-valued. Whence it

follows that the zeros of two oscillating solutions u_1 and u_2 of (5) may [depending upon the coefficients a_1, a_2, b_1, b_1, c and upon the initial conditions $u_1(\gamma_1)=\delta_1$, $u_1'(\gamma_1)=\delta_1'$; $u_2(\gamma_2)=\delta_2$, $u_2'(\gamma_2)=\delta_2'$]: (1st), separate singly or coincide as in the theory of linear equations; (2nd), separate by pairs; and (3rd), as an intermediate case, alternately separate and coincide. (The figure explains the terminology employed).

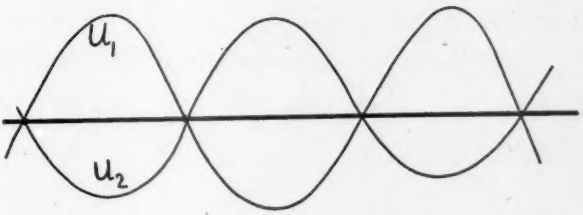
Separation.



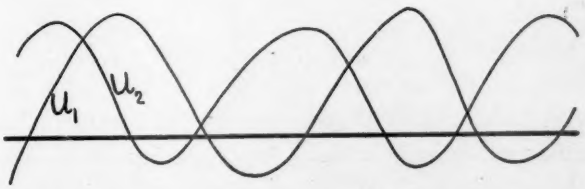
Coincidence.
(Linearly dependent solutions)



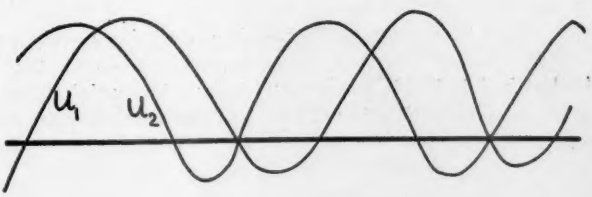
Coincidence.
(Linearly independent solutions)



Separation by pairs.



Alternate separation and coincidence.



4. *The Case of Variable Coefficients.* We begin the discussion by establishing the following

THEOREM VIII. *Consider the equation*

$$u'' + p_1 u' + p_2 u + q_1 \mid u' \mid + q_2 \mid u \mid = 0 \tag{1'}$$

*where p_1, p_2, q_1, q_2 are real, single-valued and continuous in an interval (a, b). If neither $(p_2 + q_2)$ nor $(p_2 - q_2)$ is zero in (a, b), then between consecutive zeros of u' there is one * of u.*

Assume the contrary and consider first the case where u is positive and $(p_2 + q_2) > 0$. Now at points x_1, x_2 where $u' = 0$, our equation becomes $u'' = -(p_2 + q_2)u$ and u'' is negative; the curve is therefore concave downward. There must exist, however, a point $\bar{x}$ between x_1, x_2 where u' is zero because of the fact that u' is continuous and goes from minus to plus. But the differential equation is not satisfied by such a point for we have $(p_2 + q_2) > 0$, $u'(\bar{x}) = 0$, $u(\bar{x}) > 0$, and hence $u''(\bar{x})$ is positive. Similar arguments for the cases where u is positive and $(p_2 + q_2) < 0$, and where u is negative, $(p_2 - q_2) \gtrless 0$ will each lead to a contradiction from which the theorem follows. The corresponding theorem in the linear theory † appears as a special case when $q_1 \equiv q_2 \equiv 0$. Also it is of interest to note that if we remove the non-vanishing condition imposed on the quantities $(p_2 + q_2)$, $(p_2 - q_2)$, say from the latter, then the theorem will hold for u positive but not for u negative.

We shall now consider equation (1') written in the form

$$d(Ku')/dx - L \mid u' \mid - N \mid u \mid - Gu = 0. \tag{1''}$$

We shall have need to consider also the associated equations

$$\begin{aligned}
&(2_1'') \quad d(Ku')/dx - Lu' - Nu - Gu = 0 \\
&(2_2'') \quad d(Ku')/dx + Lu' - Nu - Gu = 0 \\
&(2_3'') \quad d(Ku')/dx + Lu' + Nu - Gu = 0 \\
&(2_4'') \quad d(Ku')/dx - Lu' + Nu - Gu = 0.
\end{aligned} \tag{2''}$$

Let u_1 be a solution of an equation of the form (1'') with coefficients K_1, L_1, N_1, G_1; and similarly let u_2 be a solution of an equation of the form (1'') with coefficients K_2, L_2, N_2, G_2; from which it follows that

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* Rolle's Theorem would prohibit there being more than one.

† The author obtained this theorem from the lectures of R. G. D. Richardson, who has had it in his notes on "Linear Differential Equations of the Second Order" for years. The author is not aware of its presence elsewhere in the literature.

$$(11) \qquad d(K_1u_1')/dx - L_1 \mid u_1' \mid - N_1 \mid u_1 \mid - G_1u_1 = 0$$
$$(12) \qquad d(K_2u_2')/dx - L_2 \mid u_2' \mid - N_2 \mid u_2 \mid - G_2u_2 = 0.$$

Multiplying (11) by u_2 and (12) by $-u_1$ and adding we get, after simplifying by means of the equation themselves,

$$(13) \quad d(K_1u_1'u_2 - K_2u_2'u_1)/dx = (G_1 - G_2)u_1u_2 + (K_1 - K_2)u_1'u_2' + L_1 \mid u_1' \mid u_2 - L_2 \mid u_2' \mid u_1 + N_1 \mid u_1 \mid u_2 - N_2 \mid u_2 \mid u_1.$$

From this the following identity is readily obtained:

$$(14) \quad d[(u_1/u_2)(K_1u_1'u_2 - K_2u_2'u_1)]/dx = (G_1 - G_2)u_1^2 + (K_1 - K_2)u_1'^2 + K_2[(u_1'u_2 - u_2'u_1)/u_2]^2 + u_1^2[L_1(\mid u_1' \mid/u_1) - L_2(\mid u_2' \mid/u_2)] + u_1^2[N_1(\mid u_1 \mid/u_1) - N_2(\mid u_2 \mid/u_2)].$$

This is a generalization of the Picone identity * in the linear theory and reduces to that in the event $L \equiv N \equiv 0$.

Of two functions $y_1(x)$, $y_2(x)$, continuous in an interval (a, b), $y_2(x)$ will be said to oscillate more rapidly in (a, b) if it has more zeros than $y_1(x)$ in that interval. Let us now consider the question of the rapidity of the oscillation of the zeros of u_1 and u_2 (solutions of the two equations (11) and (12) respectively) under the following conditions on the coefficients: L_1, L_2, N_1, N_2 are non-negative; $K_1 \geq K_2 > 0$, $G_1 \geq G_2$; not all of the quantities L_1, L_2, N_1, N_2, G_1, G_2, are identically zero simultaneously. Let x_1, x_2 be consecutive zeros of u_1 which is taken to be positive between x_1, x_2. We can now state the following

THEOREM IX. *Any solution u_2 of* (12) *which is negative at some point γ: $x_1 \leq \gamma \leq x_2$, vanishes at least once in the open interval (x_1, x_2).*

Proof. Suppose, first, that $u_2(x_1) \neq 0$, $u_2(x_2) \neq 0$. Now $u_2(\gamma) < 0$ and $u_2(\xi)$, $x_1 \leq \xi \leq x_2$ will remain negative unless it vanishes in (x_1, x_2). Assuming that u_2 does not vanish, we integrate (14) between the limits x_1, x_2. The left-hand side yields zero. On the right-hand side we have a positive quantity in view of the conditions on the coefficients. The contradiction establishes the existence of at least one zero of u_2 in (x_1, x_2). Suppose, next, that u_2 vanishes at one or both of the end points of the interval considered. Even in this case it is true that u_2 has at least one zero in the open interval (x_1, x_2); for then at x_1 and at x_2 the determinate quantity u_1'/u_2' would

* See, for instance, Ince, "Ordinary Differential Equations," p. 226.

replace the indeterminate one u_1/u_2 in the left-hand member of (14) and an integration between the limits x_1, x_2 would give the desired contradiction. Thus, in (x_1, x_2), u_2 oscillates more rapidly than u_1.

If u_1 and u_2 are solutions of the same equation (1″), then equation (14) becomes, as a special case of considerable importance:

$$(15) \quad d[(u_1/u_2)K(u_1'u_2 - u_2'u_1)]/dx = K[(u_1'u_2 - u_2'u_1)/u_2]^2 + u_1^2L(|u_1'|/u_1 - |u_2'|/u_2) + u_1^2N(|u_1|/u_1 - |u_2|/u_2).$$

THEOREM X. *Given equation* (1″) *with the following conditions on the coefficients:* L, N *non-negative;* $K > 0$. *We consider a solution* u_1 *supposed positive between its consecutive zeros* x_1, x_2. *Then, a necessary and sufficient condition for another solution* u_2 *to vanish twice and only twice in the closed interval* (x_1, x_2) *is that the wronskian* $W(u_1, u_2)$ *be zero at some point* γ: $x_1 \leq \gamma \leq x_2$.

Proof of necessity. We first note that if u_2 is positive in (x_1, x_2) and vanishes at x_1 and x_2, then $W \equiv 0$ and the solutions are linearly dependent, and second that u_2 can not otherwise vanish twice in (x_1, x_2) and be positive between its zeros; for then the wronskian of two linearly independent solutions of one and the same associated linear equation [either $(2_1'')$ or $(2_2'')$] would have to be zero at some point in (x_1, x_2) and that is impossible. Hence if u_2 is to vanish twice in the interior of (x_1, x_2), it must be negative between its consecutive zeros. We have, then, u_2 vanishing twice in (x_1, x_2) and negative in between its two zeros, say γ_1, γ_2. If $\gamma_1 = x_1$, then $W[u_1(x_1), u_2(x_1)] = 0$. Similarly for $\gamma_2 = x_2$. And if $\gamma_1 = x_1$, $\gamma_2 = x_2$, x_1 and x_2 are simultaneous consecutive zeros of solutions u_1, u_2 which are in general linearly independent. This possibility we noted in the case of constant coefficients. Finally we take $\gamma_1 \neq x_1$, and $\gamma_2 \neq x_2$ with $u_2(\xi) < 0$, $\gamma_1 < \xi < \gamma_2$, and examine the ratios u_1'/u_1, u_2'/u_2. The former passes continuously through all real values from plus infinity to minus infinity between x_1, x_2 while the latter passes continuously through all real values from plus infinity to minus infinity in the sub-interval (γ_1, γ_2). These ratios are, consequently, equal at some point in (x_1, x_2) which implies that the wronskian vanishes. This completes the first part of the proof.

Proof of sufficiency. Let the wronskian be zero at some point γ. We have already seen that if u_2 is positive, then $W \equiv 0$ and the solutions are linearly dependent. Discarding this case, we see that u_2 is negative at the point γ and will remain negative throughout (x_1, x_2) unless it vanishes in that interval. We suppose that u_2 is negative throughout (x_1, x_2) and fix at-

tention on the identiy (15). Integrating (15) between the limits x_1 and γ, we have zero on the left-hand side and, because of the hypotheses on the coefficients, plus on the right-hand side which contradiction insures a zero of u_2 between x_1 and γ, say at γ_1. Similarly integrating between γ and x_2, we obtain another zero γ_2 of u_2. Finally, if γ_1 coincides with x_1 or if γ_2 coincides with x_2, an integration between x_1 and x_2 as limits will bring about the desired contradiction. That u_2 can have no zeros other than γ_1 and γ_2 we have already noted. The proof is therefore complete.

As immediate consequences of Theorem X, we set off the following corollaries:

COROLLARY I. *No solution u_2 can vanish twice in the open interval (x_1, x_2) and be positive between x_1, x_2.*

COROLLARY II. *If $W[u_1(\gamma), u_2(\gamma)] = 0$, $\gamma \neq x_1$, $\gamma \neq x_2$ and $W \not\equiv 0$, then $u_2(\gamma) < 0$ necessarily.*

COROLLARY III. *There exists a single infinitude of solutions u_2 each of which, in the interval of consecutive zeros x_1, x_2 of a given solution u_1, vanish twice and only twice.*

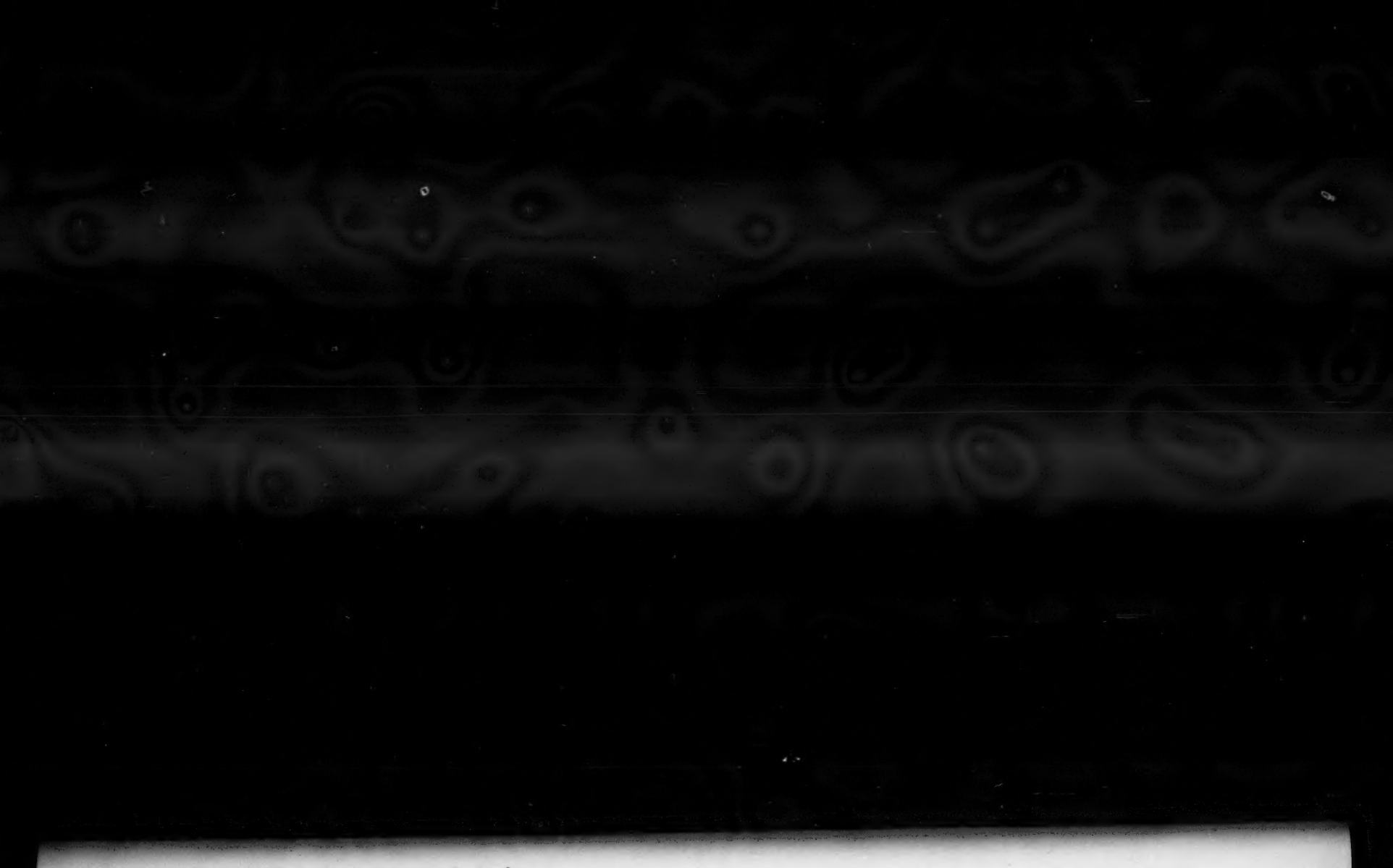
COROLLARY IV. *No solution u_2 can vanish more than twice in the closed interval (x_1, x_2); that is, separation is at most by pairs.*

The information gained from and the arguments used in the previous theorem are sufficient to establish the following

THEOREM XI. *Under the hypotheses of Theorem X, a necessary and sufficient condition for a solution u_2 to vanish once and only once in the open interval (x_1, x_2) is that the wronskian $W(u_1, u_2)$ be zero at no point in (x_1, x_2).**

Similar theorems could be developed for u_1 negative and u_2 positive by changing the conditions on the coefficients. The non-symmetric character of identities (14) and (15), however, suggests that it would be impossible to interchange the role of u_1 and u_2 and at the same time keep one and the same set of hypotheses on the coefficients. And it should be pointed out explicitly that, in the event separation is by pairs, say, as is the case in Theorem X, this separation by pairs does not necessarily persist either to the right or to the left of the interval (x_1, x_2). The same could be said about single separation. As an illustrative example, let us suppose that, in the interval (x_1, x_2), separation is by pairs u_2 vanishing at γ_1 and at γ_2 and fix our attention upon

* This becomes the separation theorem in the linear theory if $L \equiv N \equiv 0$.

a small interval Δx beginning at γ_2 and extending to the right. It is clear that the zeros of u_1 and u_2 will separate singly and mutually beyond Δx, if, in that interval, L, N pass to zero and remain identically zero to the right. As a matter of fact this is a more stringent condition than is necessary on the coefficients. Single separation would result to the right of Δx if the coefficients were changed continuously to constant values of such sort that there would result an equality of distance between consecutive zeros. And there would be other conditions on the coefficients which would necessarily bring about alternate separation of the zeros of u_1 and u_2.

Or it would be possible to pass from partial separation (i. e., alternate coincidence and separation) to single separation by the same type of changes on the coefficients. As a matter of fact, it is possible, in going from left to right, to pass from any one type of separation to any other with the exception, of course, of the case of coinciding zeros and linearly dependent solutions.